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Always Possible Frontiers

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ABSTRACT

This note addresses the issue of impossible mean-variance frontiers – those on which there are no portfolios all of whose weights are positive. This is a concern because the market portfolio should be mean-variance efficient, and it has entirely positive weights. It has been argued that impossible frontiers are likely and occur with probability one as the number of assets increases. This criticism, however, is purely statistical in nature. Similar analysis shows that arbitrage is also ubiquitous. When we recognize that the returns available in the market are not randomly distributed, but are the consequence of equilibrium process, then both of these concerns vanish. Possible frontiers will always arise.

Keywords: CAPM; Mean-Variance Analysis, Portfolio Optimization, Roll Critique, Shortselling, Long/Short.

JEL Codes: G11, G12

In a recent paper, Brennan and Lo (2010), henceforth BL, have argued that impossible mean-variance frontiers are almost inevitable in markets with a large number of assets. They define an impossible efficient frontier as one on which no point has a positive portfolio weight for every asset, and therefore no point can represent the market portfolio. They demonstrate that impossible frontiers are very likely, and, indeed, occur with probability one in the limit as the number of assets grows. They also argue that the available data indicate that observed frontiers are also empirically impossible. Levy and Roll (2015) and Brennan and Lo (2015) have argued about the empirical issue. This note addresses their theoretical result.

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This note makes two points. First, there is nothing unusual about the impossibility of efficient frontiers in large random economies. Reasoning similar to that of BL shows that simple arbitrage is also inevitable in large random economies. Second, it argues that both these results are due to the scope of random economies considered. When prices, and therefore returns, are allowed to adjust as they do in equilibrium, the impossibilities vanish. Arbitrages disappear, and, likewise, efficient frontiers become possible.

1 Inevitable Arbitrage

BL consider sequences of economies with an increasing number of assets and show that if the covariance matrix is selected at random, the sequence leads to an impossible frontier with probability one. Here I answer the following unrelated though analogous questions. What is the probability that there is no arbitrage in a complete-market economy with n assets? And how does this probability change as n grows?

It is well known that any complete market has a unique state price vector, $\mathbf{q}$, relating the matrix of payoffs, $\mathbf{X}$, to the vector of initial prices, $\mathbf{p}$,

$$\mathbf{p} = \mathbf{X}\mathbf{q}, \tag{1}$$

and that the absence of arbitrage is equivalent to the state price vector being entirely positive so there is no arbitrage if and only if $\mathbf{X}^{-1}\mathbf{p} = \mathbf{q} \gg \mathbf{0}$. How likely is this to be true in a random economy? The answer to this question depends, of course, on how the random economy is selected.

Following BL's assumption for the covariance matrix, fix $\mathbf{p}$ and assume for $\mathbf{X}$ a diffuse prior over all full-rank payoff matrices so that any one such matrix is just as likely as another.[1] Now consider any payoff matrix that admits no arbitrage so $\mathbf{X}^{-1}\mathbf{v} = \mathbf{q} \gg \mathbf{0}$. The matrix $\hat{\mathbf{X}}$, which is identical to $\mathbf{X}$ except that its j^{th} column is replaced by the negative of the j^{th} column of $\mathbf{X}$, will generate exactly the same state prices except that $\hat{q}_j = -q_j < 0$. But by assumption, the two matrices are equally likely so the probability

[1]The space of all invertible matrices is unbounded so a uniform prior has infinite mass and is therefore improper. BL spend some time discussing how to remedy this problem for positive definite matrices. Similar arguments could be made here, but this point is not the thrust of this note so it can simply be assumed that the payoff matrices are selected from a bounded subspace for which a prior is well-defined. For example, the payoff matrices can be selected randomly from the space of all invertible matrices with $\|\mathbf{X}\| \equiv \max_{ij} |X_{ij}| < \bar{X}$.

that the q_j is negative for a random payoff matrix must be 50%. This reasoning applies to each of the state prices separately so the probability that there are no negative state prices and therefore no arbitrage in a random complete market economy with n assets is no more than 2^{-n}. Thus, in the limit, arbitrage is certain in economies with a large number of assets.

This analysis leads to an even more unpalatable conclusion than BL's ubiquitous impossible mean-variance frontiers, but the spirit of the argument is exactly the same. What is missing in both cases is any economic restriction on the payoff or return matrices. Economic conditions clearly make some payoff matrices more likely than others. For example, payoffs tend to be positively correlated while limited liability ensures that some elements of $\mathbf{X}$ are positive. Either of these restrictions would tend to make one of the payoff matrices, $\mathbf{X}$ or $\hat{\mathbf{X}}$, more likely than the other. Yet the most obvious restriction is that $\mathbf{p}$ and $\mathbf{X}$ are not randomly related but instead the values are determined in equilibrium. Recognizing this single condition completely eliminates any arbitrage. For any *arbitrary* non-singular payoff matrix, there is always a vector of initial values for which all the state prices are positive and arbitrage is excluded—in fact there is a value vector for *any* positive state price vector, $\mathbf{q}$, namely $\mathbf{p}(\mathbf{q}) = \mathbf{X}\mathbf{q}$. Price adjustment can assure that there is no arbitrage. This must be true in a resulting equilibrium, and remains true even if prices don't adjust fully to their equilibrium values.

2 Always Possible Frontiers

When initial prices are permitted to adjust, the same basic rescue occurs for mean-variance impossibility. For *any* arbitrary covariance matrix and expectations for the end-of-period random payoffs, there are always beginning-of-period prices so that the mean-variance efficient frontier of returns[2] has

[2]Traditionally, only the upper limb of the minimum-variance frontier has been considered efficient; however, portions of the lower limb can also be optimal portfolios. A two-asset example is given on page 112 of Ingersoll (1987). This can be extended to multiple assets using the Ross (1978) separating distributions. While this distinction could be relevant in BL's definition of impossible frontiers, it is immaterial in this note's demonstration of possibility. Nevertheless, for consistency, the upper-limb-only definition is used here.

entirely positive share holdings in some region.[3] In fact, there are initial prices that can put *any* set of holdings onto the efficient frontier.

To find these price vectors, the problem is stated in payoffs rather than in the endogenous returns. Let $\bar{\mathbf{x}}$ and $\boldsymbol{\Xi}$ be the expectation and covariance matrix of the payoffs per share. These are completely arbitrary except that $\boldsymbol{\Xi}$ is assumed to be positive definite to eliminate risk-free arbitrage. An arbitrary share-holding vector, $\mathbf{h}$, is a portfolio on the mean-variance frontier if the initial prices satisfy

$$\mathbf{p} = \frac{\bar{\mathbf{x}} - a\boldsymbol{\Xi}\mathbf{h}}{b} \tag{2}$$

for some positive values for a and b.[4]

Define the matrix $\mathbf{P} = \mathbf{diag}(\mathbf{p})$ as the diagonal matrix whose i^{th} diagonal element is p_i as given in (2). Then the vector of expected returns per dollar and the covariance matrix of returns per dollar are

$$\boldsymbol{\mu} = \mathbf{P}^{-1}\bar{\mathbf{x}} \quad \boldsymbol{\Sigma} = \mathbf{P}^{-1}\boldsymbol{\Xi}\mathbf{P}^{-1}. \tag{3}$$

To verify the claim that $\mathbf{p}$ leads to a possible frontier including $\mathbf{h}$, consider the frontier portfolio whose zero-beta return per dollar is equal to b—this would be the tangency portfolio if the risk-free return were b. This

[3] Care is required in defining a possible frontier. BL define it as one with a segment in which every asset's *portfolio weight* is positive, but this definition is valid only if the initial prices of all assets are positive. A proper definition of a possible frontier is one with some segment in which *the number of shares held* of every asset is positive. These are not the same because an asset with a negative price has a negative portfolio weight when a positive number of shares are held. The usual assumption of limited liability ensures that prices are positive in a standard model which would make these two definitions equivalent. However, positive initial prices cannot be assured under mean-variance analysis, even with limited liability. Very high payoffs increase an asset's variance more than they increase its expected payoff, and, as indicated in (2), this can lead to negative prices even when assets have limited liability. Positive prices must be imposed separately as an assumption or determined indirectly. See Dybvig and Ingersoll (1982) for further discussion about the effects of a large upper tail on the mean-variance problem.

[4] The price vector in (2) will be recognized as the multivariate version of the valuation result in Jensen and Long, Jr. (1972). In their analysis, a and b are a measure of risk aversion and one plus the risk-free interest rate. Here they are free parameters. The portfolio $\mathbf{h}$ will be on the efficient frontier for any positive a. Assuming that expected payoffs are positive, a can always be chosen small enough so that all prices are positive. This ensures that returns are meaningfully defined and that portfolios with positive demands have positive portfolio weights.

portfolio's weights are $\boldsymbol{\omega} = c\,\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - b\,\mathbf{1})$ where $\mathbf{1}$ is a vector of 1's, and $c \equiv [\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - b\,\mathbf{1})]^{-1}$ is the normalizing constant. The weights, $\boldsymbol{\omega}$. and number of shares, $\mathbf{n}$, in any portfolio are related by $\boldsymbol{\omega} = \mathbf{P}\mathbf{n}/v_p$, where v_p is the initial value of the portfolio. So the share demands of this portfolio if it has an initial cost of $v_p = \mathbf{h}'\mathbf{p}$ are

$$\begin{aligned} \mathbf{n} &= (\mathbf{h}'\mathbf{p})\mathbf{P}^{-1}\boldsymbol{\omega} = (\mathbf{h}'\mathbf{p})\,c\,\mathbf{P}^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - b\,\mathbf{1}) \\ &= (\mathbf{h}'\mathbf{p})\,c\,\mathbf{P}^{-1}\mathbf{P}\boldsymbol{\Xi}^{-1}\mathbf{P}(\mathbf{P}^{-1}\bar{\mathbf{x}} - b\,\mathbf{1}) \\ &= (\mathbf{h}'\mathbf{p})\,c\,\boldsymbol{\Xi}^{-1}(\bar{\mathbf{x}} - b\mathbf{P}) = (\mathbf{h}'\mathbf{p})\,c\,a\,\boldsymbol{\Xi}^{-1}\boldsymbol{\Xi}\mathbf{h} = (\mathbf{h}'\mathbf{p})\,c\,a\,\mathbf{h}. \end{aligned} \tag{4}$$

The third equality follows from the definitions in (3). The last equality comes from (2) and $\mathbf{p} = \mathbf{P}\mathbf{1}$. The latter two relations also set the parameter c to be

$$\begin{aligned} c &= \left[\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - b\,\mathbf{1})\right]^{-1} = \left[\mathbf{1}'\mathbf{P}\boldsymbol{\Xi}^{-1}\mathbf{P}(\mathbf{P}^{-1}\bar{\mathbf{x}} - b\,\mathbf{1})\right]^{-1} \\ &= [\mathbf{1}'\mathbf{P}\boldsymbol{\Xi}^{-1}(\bar{\mathbf{x}} - b\mathbf{p})]^{-1} = (a\mathbf{p}'\mathbf{h})^{-1}. \end{aligned} \tag{5}$$

Therefore, $\mathbf{n} = \mathbf{h}$ and this specific frontier portfolio has the desired holdings. As $\mathbf{h}$ is arbitrary, it can be chosen so that $\mathbf{h} \gg \mathbf{0}$, and we have a possible frontier.

The final requirement is that portfolio $\mathbf{h}$ is located on the upper (efficient) half of the hyperbola. This will be true whenever b is less than the expected return on the global minimum-variance portfolio, that is, $0 < \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - b = c/(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})$. The bilinear form, c, was evaluated in (5). It is positive if the amount invested in the portfolio, $\mathbf{p}'\mathbf{h}$, is positive. This can be guaranteed for positive holdings if all prices are positive, which can be assured as noted in footnote 4.

Furthermore, because the holdings vary continuously along the frontier, they will all be positive within some interval surrounding this point as well. This analysis applies for any $\mathbf{h}$. Of course, for an equilibrium we would set $\mathbf{h}$ to the aggregate supplies.

The analysis here has assumed a financial market in which equilibrium is achieved through price adjustment. In such a model, rates of return obviously cannot be exogenously specified, and as shown, prices can always adjust so that demand will equal the positive supply. Conversely, in a model of real investment, rates of return can sometimes be assumed exogenous. For example, if all projects have stochastic constant returns to scale. In such a model, there might be impossible frontiers when all projects are considered. However, equilibrium will now arise through adjustment in supply. Those projects for which the demand would be negative will not be undertaken. The remaining projects with positive investment will naturally form a possible frontier (assuming that the mean-variance criterion is used in evaluations).

3 Conclusion

This note has demonstrated that impossible mean-variance frontiers, those composed entirely of portfolios with some short positions, are a mathematical curiosity rather than an economic concern. Similar reasoning proves that arbitrage opportunities must always be available, yet clearly they are not. Mathematical reasoning is a powerful tool, but it has its limits. Uninformative priors may be a valid starting point, but priors must not be so uninformed as to be naïve.

Despite our use of static models, economics is a description of the world in constant adjustment—supplies change, prices vary, and beliefs alter. The latter two are of primary importance in the market for equities. Even if we agree that the stock market is not 100% efficient, it certainly is not as wildly inefficient as it would be if prices were set at random. Even if prices are only somewhat related to payoffs, then return distributions are not random either.

It really should come as no surprise that possible frontiers are rare. Although we use the word equilibrium in both economics and physics, the word means something quite different in each field. In physics, an equilibrium is something created by randomness, and statistical mechanics is used to study it. In economics, an equilibrium is a state brought about by purposeful actions on the parts of agents. If we ignore the economics in our models, then we get the same entropy of physics models: disorder in the form or arbitrage and impossible frontiers appear. We use Bayesian analysis and game theory in economics because we must.

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