Does Macro-Asset Pricing Matter for Corporate Finance?

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ABSTRACT

In an asset-pricing model calibrated to match the standard asset pricing empirical properties – in particular, the time-variation in the equity premium – we calculate the value implications of sub-optimal capital budgeting decisions. Specifically, we calculate that an investment policy that ignores the time variation in the equity premium, such as would occur with a cost of capital using a static CAPM-like model, incurs a 11.7% value loss. We also document the implications for a firm's asset returns in this context.

1 Introduction

Determining cost of capital is central in corporations' decisions on capital budgeting. To estimate the cost of capital, the Capital Asset Pricing Model has been the de facto standard approach in business education and practice. Almost all textbooks in corporate finance use the CAPM as a workhorse model to determine the discount rate. Consistent with the CAPM's widespread use in education, a survey by Graham and Harvey (2001) found that 73.5% of U.S. and Canada companies use it. The CAPM is, of course, a static model and is agnostic about the level of the equity premium or its dynamics. Common practice is to use a constant value for the equity premium from 4% to 7% (Welch (2000) and subsequent updates). In contrast, much macro-asset pricing analysis focuses on the time-variation in the equity premium. Cochrane (2011), for example, points out that the time-variation is on the same order of magnitude as the level; a one standard deviation change above or below the mean swings the equity premium between 11% and 1%. [1]

If the equity risk premium is time-varying, using a constant discount rate will mislead firms in capital budgeting. In this paper, we quantify the value loss caused by an investment policy that wrongly disregards time-variation in the risk premium. Specifically, we contrast the sub-optimal investment policy with the optimal investment that correctly updates the discount rate in response to the time-varying risk premium. Comparing the firm values that these two investment policies would result in, we are able to measure the value loss. Since our goal is quantitative, we address this question through a model calibrated to match the key characteristics of corporate investment and equity returns.

To motivate this quantitative analysis, we first examine whether the sub-optimality indeed exists in the empirical data – whether or not corporate investments react to swings in discount rates. This analysis, of course, requires a measure of the discount rate, but the conventional approaches such as the factor models are of little help in capturing the temporal changes in the discount rates (Welch and Goyal (2008)). Instead, we use the theoretical relation between investments and discount rate found by Kogan and Papanikolaou (2012), and we recover the discount rate that

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[1]Routledge and Zin (2010), building on Melino and Yang (2003), point out that the original Mehra and Prescott (1985) equity risk premium model implies a dynamic equity risk premium.

is implicit in the observed investments. This investment-implied discount rate should be able to predict future stock returns if firms correctly update the discount rate in response to time-varying risk premium. Contrary to the hypothesis, the regression analysis finds that the implied discount rate is insignificant for predicting the subsequent returns. We see this finding as evidence that firms may not properly update the discount rates in practice.

Next, in order to link an investment policy to firm value, we construct a dynamic model of capital investment. In the model, a firm is presented with an opportunity to invest in a new project on each date. Taking on the project requires an upfront investment. Once invested, the project delivers cash flows that are correlated with marginal utilities (e.g., aggregate consumption). A project is a now-or-never option as in Berk *et al.* (1999), so the firm invests in the project when its NPV is positive. In this setup, the NPV and investment decisions are affected by the project's systematic risk – covariance between project cash flows and consumption.

We specify the marginal utilities following the long-run risk model of Bansal and Yaron (2004). Here, the stochastic volatility of consumption growth causes the equity risk premium to fluctuate. In this framework, we consider two economies. One has a constant equity premium (from a constant volatility assumption), and the other features a dynamic equity premium. Both economies are calibrated to match the usual moments of aggregate asset returns. In each economy, we consider two representative firms and their investment policies. One firm - Type 1 - acts as if the equity premium is constant. The other - Type 2 - acts as if the equity premium is dynamic. This will let us consider the optimal investment behavior (Type 1 in the constant-volatility economy and Type 2 in the stochastic-volatility economy) as well as measure the cost of a sub-optimal policy. In particular, we measure the cost of acting as a Type 1 firm (a CAPM-like constant equity risk premium) in an economy with a dynamic equity premium. We also estimate the counter-factual cost of a Type 2 firm in an economy with a static equity premium.

Our main finding is as follows: in an economy with a dynamic equity premium, the present value of growth options for Type 1, which sub-optimally invests, is 11.7% lower than the value of Type 2. The value loss to Type 1 is due to its failure to adjust the discount rate to a fluctuating risk premium; this results in over-investment when the risk premium is high and under-investment when the risk premium is low. In contrast, if the economy features a static equity premium, the Type 2 firm incurs only

a 0.1% loss from the sub-optimal investment. The asymmetry in the value loss is driven by the timing of sub-optimal investment. In the economy with a dynamic equity premium, the Type 1 firm over-invests most at the state of highest uncertainty. This is exactly the state where the marginal rate of substitution is the highest. Hence, the sub-optimal investment is very costly. Conversely, the Type 2 firm in the economy with a static equity premium is not exposed to such coordination between erroneous investment and the marginal utilities, so it suffers only minor loss.

We find that the estimate of value loss is fairly robust to different calibrations. When the Type 1 pricing kernel and investment policy are alternatively calibrated to match the unconditional second or fourth moments of consumption growth or the price-dividend ratio to those of Type 2, the resulting loss is 6.9% - 13.6%. Similar to the baseline calibration, the value loss to the Type 2 remains minimal in the counter-factual economy with a static equity premium; the loss is 0.22% or lower across different calibrations. Besides, we document that the firm types yield statistically significant return differences in each economy. The returns on firms with sub-optimal investment policy are higher than those with correct investment in both economies.

A growing number of studies have explored the implications of the equity premium dynamics for corporate decisions. These include Bharmra *et al.* (2010), Chen (2010), and Chen *et al.* (2016). They have studied how firms' optimal capital structure responds to fluctuations in the equity premium, but they have not explicitly modeled firms' investments for the sake of simplicity. Arnold *et al.* (2013) and Chen and Manso (2017) take a step toward endogenizing the investment decisions in the presence of dynamic equity premium. For tractability, they describe firms' investment as one or a few exogenous shifts in firms' cash flows over the firm's life. However, it would be hard to calibrate this stylized investment to match the empirical data that appears as a time-series of investment expenses. Similar to Kuehn and Schmid (2014) and Gomes and Schmid (2016), we let firms make investment decisions regularly over time. We complement these studies by providing the quantitative evidence that the understanding of equity premium dynamics matters for corporate decisions.

The paper is organized as follows. In the next section, we investigate the empirical relationship between corporate investments and discount rates. In section 3, we describe the economic environment and firm-level investments. Section 4 provides the valuations of projects and then

presents the investment rule and the resulting firm values. In section 5, we calibrate the model and compare firm values and returns for different investment rules. We conclude in section 6.

2 Empirical Relation Between Corporate Investments and Discount Rates

The primary goal of the paper is to measure a value loss for a sub-optimal investment policy that ignores time-variations in the equity premium. To motivate this quantitative analysis, we first examine whether the sub-optimality indeed exists in empirical investments.

This time-series analysis requires a measure of conditional equity premium or discount rate on a certain date. However, the conventional approaches, the CAPM or other factor models, are of little help in detecting changes in discount rate, as pointed out by Welch and Goyal (2008). We, therefore, take the following alternative approach. Using the observations of investments, we recover the discount rate implicit in the investments. Then, we examine whether the implied discount rates are consistent with subsequently realized returns on stocks.

This recovery of the discount rate from the investments is based on a theoretical prediction of the neoclassical investment model. To specify, let I_t denote a firm's capital investment, K_t denote the capital stock, D_t denote the dividend amount, and R_t denote the gross return on the firm's stock. Kogan and Papanikolaou (2012) derive the association between the investment and the expected return as follows:

$$\lambda \ln (I_t/K_t) \approx \text{const} + E_t \left[\ln \frac{D_{t+1}}{K_{t+1}} + \sum_{j=1}^{\infty} \left(\rho^j \ln \left(D_{t+j+1}/D_{t+j} \right) - \rho^{j-1} \ln R_{t+j} \right) \right]$$
(1)

where λ is a parameter in investment-adjustment costs and ρ is a constant that depends on the average price-dividend ratio. The formal derivation can be found in Kogan and Papanikolaou (2012), but here is an overview. The theoretical model allows us to determine the optimal level of investment as a function of a firm's productivity and capital stock. Intuitively, these two variables also determine the firm value. Thus, the optimal investment and the firm value are interconnected, and the parametric link between the two can be found through the first-order condition of the op-

timal investment. Next, by expressing the firm value in terms of the price-dividend ratio and using Campbell and Shiller (1988) decomposition, we obtain equation (1).

Plugging data on firms' investments, capital stock, and dividend growth in equation (1), we are able to find the date-t expectation of future returns, or discount rate, that firms use in their investment decisions. Then, the null hypothesis is that this investment-implied discount rate should forecast subsequently realized returns, if firms correctly update the discount rate.

We empirically test the null hypothesis using quarterly returns on industry portfolio for the manufacturing industry during years 1972-2017. The return data is from Kenneth French's website. We choose to analyze at the portfolio-level because the firm-level investment is more likely to be subject to other forces such as financial constraints, which are not taken into account in equation (1). Details on how to measure other required variables are provided in Appendix A.1.

Having obtained the investment-implied discount rate, we regress the portfolio's realized returns on the implied discount rate. Theoretically, the dependent variable is the infinite sum of future returns, and we approximate it as the sum of realized returns for either the next five years or the next ten years. Table 1 shows the regression results. A finding is that, in contrast to the hypothesis, the implied discount rate is insignificant in predicting the future returns; the t-statistics are -0.26 for the five-years return in the specification (1) and -1.58 for the ten-years return in the specification (6).

It is possible that this insignificance arises from the time to build in capital stock as pointed out by Lamont (2000); capital expenditure on date t actually reflects investment decisions of earlier dates. This delay means that the date-t discount rate needs to be recovered from later observations of investment. We consider this possibility in specifications (2) through (5) and (7) through (10) and use the investment observed one to four quarters later than date t. In some specifications, this consideration leads to a positive coefficient on the implied discount rate as the theory predicts, but the association remains statistically weak.

To summarize, our empirical analysis reveals that the discount rates that firms use in practice for capital budgeting do not fully capture the fluctuations in the expected return. In the following sections, we set up a model and quantify the value loss that such sub-optimal investment will

cause.

3 Model

We model the pricing kernel following Bansal and Yaron (2004) endowment economy. Characterizing the pricing kernel as the product of preferences and consumption growth facilitates calibration and comparison. At the firm-level, the firm has an opportunity to invest in projects delivering cash flows that are correlated with the consumption growth. This set-up gives the projects their "systematic" risk. Our model is only partial equilibrium, since we do not connect the sum of all projects in the economy back to aggregate consumption.

3.1 Economic Environments

Preferences of the representative agent are recursive as in Epstein and Zin (1989) and Weil (1989). Preferences at date t are given by

$$U_t = \left[(1-\eta)c_t^\rho + \eta\mu(U_{t+1})^\rho \right]^{1/\rho} \qquad (2)$$

where ρ captures time preference (the intertemporal elasticity of substitution is $1/(1-\rho)$). μ is the expected utility certainty equivalent, i.e., $\mu(U_{t+1}) = E_t\left[U_{t+1}^\alpha\right]^{1/\alpha}$, where α is the risk-aversion parameter (the coefficient of relative risk aversion is $1-\alpha$). The marginal rate of substitution – the pricing kernel – is

$$m_{t+1} \quad = \quad \eta(c_{t+1}/c_t)^{\rho-1}\left[U_{t+1}/\mu(U_{t+1})\right]^{\alpha-\rho}$$

where $(c_{t+1}/c_t)^{\rho-1}$ captures the short-run consumption growth risk, and $(U_{t+1}/\mu_t(U_{t+1}))^{\alpha-\rho}$ captures the innovation to continuation utility and reflects the long-run risk.

The consumption growth from date $t-1$ to t, denoted by $g_t = c_t/c_{t-1}$, is described with the underlying state variable, x_t, a vector of arbitrary dimension. Specifically, the logarithm of consumption growth is assumed to be $\log g_t = g + e^T x_t$, where e is a constant vector. The dynamics of the state variable x_t features AR(1) with a stochastic volatility:

$$
\begin{aligned}
x_{t+1} &= Ax_t + v_t^{1/2}Bw_{t+1} \\
v_{t+1} &= (1-\varphi)v + \varphi v_t + bw_{t+1}
\end{aligned}
\qquad (3)
$$

where v is the unconditional mean of v_t, w_t is a vector of independent normal random variables with zero mean and the covariance equal to the identity matrix, and $Bb^T = 0$. In this expression, x_t and v_t control the conditional mean and volatility of consumption growth in future and summarize the state of economy.[2] The stochastic volatility in the growth is the source of the creation of time-variation in the equity premium. Therefore, we can represent the different types of economy or firm – having the dynamic equity premium or not – by turning on or off the stochastic volatility channel. Thus, to describe the economy with constant equity premium, we can impose that the volatility of consumption growth is constant.

This process for consumption growth implies that the pricing kernel is

$$\log m_{t+1} \;=\; \delta_0 + \delta_x^T x_t + \delta_v v_t + \lambda_x^T w_{t+1} + \lambda_v^T w_{t+1} \tag{4}$$

where δ_0, δ_x, δ_v, λ_x, and λ_v are known functions of preference and consumption dynamics parameters. The derivation of the pricing kernel is provided in Appendix A.2 through A.4. This pricing kernel determines the dynamic properties of the equity risk premium. (In particular, see the equation (A.18)). Key to the dynamic properties is the time-variation in the volatility of consumption growth. If v_t is constant ($b = 0$ in equation (3)), then the equity risk premium is constant.

3.2 Firms

The description of firms' investment is similar to Berk *et al.* (1999). Firms operate with an infinite horizon. For each date, a new opportunity becomes available to a firm. The firm decides whether or not to undertake the new project. If the project is not accepted, that opportunity disappears (a now-or-never option), and the firm will receive another new opportunity next date. If the firm decides to invest, the initial cash flow is negative (investment) followed by subsequent positive cash flows. Project termination is finite and deterministic in our setting.

Undertaking a project at date t requires an upfront investment of I. Once undertaken, the project delivers positive cash flows that are corre-

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[2]If $e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} \phi & \theta \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & \sigma_w \end{bmatrix}$, the dynamics are an approximation of Bansal and Yaron (2004)'s consumption dynamics with stochastic volatility. The Gaussian shock to volatility is an approximation, obviously. It is straightforward to relax but makes the algebra less transparent.

lated with the consumption growth. Let d_{t+s} denote cash flow at date $t+s$ from the project. The growth in cash flow at date $t+s$ is given by

$$\frac{d_{t+s}}{d_{t+s-1}} = \exp\left(g + e^T(Ax_{t+s-1} + \beta_t v_{t+s-1}^{1/2}Bw_{t+s}) - \frac{\beta_t^2 v_{t+s-1}}{2}e^TBB^Te\right). \quad (5)$$

where β_t controls the covariance between the cash flows and consumption, i.e., the systematic risk of the project. The basic idea of the expression for cash flow is that the mean growth and volatility of the project-level cash flows are influenced by the economic state x_t and v_t, which describe consumption growth. Specifically, the growth in project cash flow is the mean-preserving spread of the consumption growth, where β_t determines the degree of the spread. The parameter for the systematic risk, β_t, is specific to a project. It is drawn from a uniform distribution $[0; \beta_{\max}]$. The firm observes a realized value of the project-level systematic risk and makes the investment decision. Once realized, β_t is constant for the life of the project. The project generates cash flows for N periods and zero afterwards. For all projects, we normalize the size of investment, I, to be 1, and the first cash flow of the project $d_t = d^0$.

This specification of project might seem to imply a strong tie between project payouts (dividends) and aggregate consumption. However, at the firm-level, the specification still produces an imperfect correlation between the firm's operating income and consumption. Firm-level income is a collection of cash flows of many projects that have different realizations of project-level systematic risk from each other. This idiosyncrasy in exposure to the aggregate risk produces a loose link between the firm payout and consumption, consistent with data.

4 Valuation

To determine optimal investment policy, we calculate the value of a potential investment (NPV). To do that, we start by valuing a single cash flow. From there, we calculate the value of assets-in-place, optimal investment, and the value of growth options.

4.1 Value a Project

A project, once in place, is a collection of risky cash flows, $d_t, ..., d_{t+N}$. Their correlation with aggregate consumption is determined by the pa-

rameter β_t. Consider one of these cash flows, d_{t+s}. Define the date-t value of this single cash flow as $q_t^s d_t$, where q_t^s is the price-to-cash-flow ratio (or, more loosely, the "price-dividend" ratio). We calculate these prices recursively. The date-t price-to-cash-flow ratio of the asset that pays at date $t+1$ is determined by

$$q_t^1 = E_t \left[m_{t+1} \frac{d_{t+1}}{d_t} \right]. \tag{6}$$

Similar to the prices of "zero-coupon equity" in Lettau and Wachter (2007), the price is an exponential affine function of the state variables as follows:

$$q_t^1 = \exp\left(\delta_0 + g + \frac{\lambda_v^T \lambda_v}{2} + (\delta_x^T + e^T A)x_t + \left(\delta_v + \frac{\lambda_x^T \lambda_x^T}{2} + \beta_t e^T B \lambda_x \right) v_t \right) \tag{7}$$

where δ_0, δ_x, δ_v, λ_x, and λ_v are the parameters in the pricing kernel. The price-to-cash-flow ratio of the asset with maturity $s > 1$ is

$$q_t^s = \exp\left(D_{0,s} + D_{x,s} x_t + D_{v,s} v_t \right) \tag{8}$$

where $D_{0,s}$, $D_{x,s}$, and $D_{v,s}$ are constants recursively related to the constants for the asset with maturity of $s-1$ via the following:

$$\begin{array}{rcl} D_{0,s} & = & \delta_0 + g + D_{0,s-1} + D_{v,s-1}(1-\varphi)v + (1/2)\left(\lambda_v^T + D_{v,s-1}b\right)\left(\lambda_v^T + D_{v,s-1}b\right)^T \\ D_{x,s} & = & \delta_x^T + (e^T + D_{x,s-1})A \\ D_{v,s} & = & \delta_v + D_{v,s-1}\phi_v - (1/2)\beta_t^2 e^T BB^T e + (1/2)(\lambda_x^T + \beta_t e^T B + D_{x,s-1}B)(\lambda_x^T + \beta e^T B + D_{x,s-1}B)^T. \end{array} \tag{9}$$

The derivation is provided in Appendix A.6.

4.2 Value Assets in Place

A firm has two sources of value – assets-in-place and growth options. Assets-in-place are a collection of existing projects coming from past investment decisions. Growth options denote future investment opportunities.

The NPV of the date-t project, which is denoted by p_t, is

$$p_t = E_t \left[\sum_{s=0}^{N} m_{t,t+s} d_{t+s} \right] - I \tag{10}$$

where $m_{t,t+s} = \prod_{k=1}^{s} m_{t+k}$. Using the prices of elementary assets as de-
fined above, the NPV is

$$p_t = d^0 \cdot E_t \left[\sum_{s=0}^{N} m_{t,t+s} \frac{d_{t+s}}{d_0} \right] - I = d^0 \cdot \sum_{s=0}^{N} q^s(x_t, v_t, \beta_t) - I. \qquad (11)$$

As the investment opportunity is a now-or-never option, the firm under-
takes a project whenever its NPV is positive. The firm's investment deci-
sion at date t depends on both the project-level systematic risk, β_t, and
the state of the economy, (x_t, v_t). Notice that investment policy also de-
pends on the pricing kernel that determines cash flow values. Below, we
will consider firms that differ in their model of the pricing kernel.

We represent the investment decision at date j with an indicator χ_j
such that $\chi_j = 1$ if the firm invests or 0 otherwise. The value of-assets-in
place, denoted by K_t, is

$$K_t = \sum_{j=t-N+1}^{t} \sum_{s=1}^{N-t+j} \chi_j d_t^j q^s(x_t, v_t, \beta_j)$$

where d_t^j is date-t cash flow of project that was invested at date j. Note
the equation is simplified by the assumption that project life is N periods.

4.3 Value Growth Options

A firm's growth options is the collection of all investment opportunities
that the firm will have in the future. To determine the value of growth op-
tions, we first consider a single investment opportunity available at $t + 1$.
Because a firm will take on the project only if its NPV turns out to be posi-
tive, the payoff of the investment opportunity is similar to that of a finan-
cial option. When both project-level systematic risk and economic states
are realized at date $t+1$, the option value is $\max(p_{t+1}, 0)$. Let $f(x_{t+1}, v_{t+1})$
denote the value of the option conditional on states (x_{t+1}, v_{t+1}) and prior
to realization of project-specific risk β_{t+1}. It follows that the option value
is given by

$$f(x_{t+1}, v_{t+1}) = \int_0^{\bar{\beta}(x_{t+1}, v_{t+1})} p(x_{t+1}, v_{t+1}, \beta) \frac{1}{\beta_{max}} d\beta \qquad (12)$$

where $\overline{\beta}(x_{t+1}, v_{t+1})$ is the investment threshold of the project's systematic risk such that $p(x_{t+1}, v_{t+1}, \overline{\beta}) = 0$. Note that the firm invests only when the systematic risk is lower than the threshold. This threshold in turn depends on the current economic state (x_{t+1}, v_{t+1}).

The firm has a series of investment options which will become available from $t + 1$ onward. The date-t present value of growth options, denoted by $S(x_t, v_t)$, can be expressed in a recursive way:

$$
\begin{aligned}
S(x_t, v_t) &= E_t \left[\sum_{s=1}^{\infty} m_{t,t+s} \max(p_{t+s}, 0) \right] \\
&= E_t \left[m_{t,t+1} E_{\beta_{t+1}} \left[\max(p_{t+1}, 0) \right] + m_{t,t+1} E_{t+1} \left[\sum_{s=1}^{\infty} m_{t+1,t+1+s} \max(p_{t+1+s}, 0) \right] \right] \\
&= E_t \left[m_{t,t+1} f(x_{t+1}, v_{t+1}) + m_{t,t+1} S(x_{t+1}, v_{t+1}) \right].
\end{aligned}
\tag{13}
$$

The expectations operator, $E_{\beta_{t+1}}$, denotes the expectation over the distribution of systematic risk β_{t+1}. In the derivation, we use the law of iterated expectation to value the investment option available at date $t + 1$. From the recursive structure, the present value of growth options is solved numerically as a function of the state variables.[3]

Using the expression of growth options, we can look into the impact of using an incorrect pricing kernel. Suppose a firm has an incorrect model of the pricing kernel – say, it ignores the dynamic properties of the risk premium. Then, its investment policy is going to be sub-optimal. Accordingly, its value for the single option, $f(x, v)$, is lower than what the optimal policy would have. Moreover, the degree of the sub-optimality can be state-dependent. If the firm tends to invest particularly poorly when the marginal rate of substitution is high, the value loss can be large.

4.4 Calibration

We are interested in the quantitative implications of dynamic equity premium on a firm's investment policy. Therefore we need a sensibly calibrated model. Here, we look at two economies. One economy has a constant risk premium and the other with a dynamic risk premium. Both are set to match the empirical moments of consumption growth and aggregate asset returns, including the equity premium. In each, the project characteristics are chosen to reproduce the empirical facts on firms' asset and

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[3]For the computation, we use a finite support of state variables by discretizing the support of the state based on the method of Tauchen (1986). We then solve for growth options with the value function iteration.

income characteristics. Note that we calibrate each model separately. To generate a dynamic equity premium, we calibrate a model with stochastic volatility in consumption growth. Then, we create a static equity premium model by simply turning off the stochastic volatility (set $b = 0$ in equation (3)). However, that produces an economy that does not match data. Instead, by making the two economies differ only as to whether the equity premium is dynamic or static and have otherwise the same empirical properties, we can fairly compare these two economies and isolate the impact of the equity premium dynamics. Table 2 shows the calibration parameters and results.

We calibrate the model in two steps. First, each economic environment is calibrated to fit the stylized facts of consumption growth and aggregate asset returns, using the empirical moments in Bansal and Yaron (2004). Following their calibration method, we simulate 1,000 samples of 840 month-long time series of consumption growth, given a choice of parameters characterizing consumption and preferences. In matching aggregate stock returns, we regard the consumption stream as the aggregate equity and compute its monthly returns. Appendix A.5 provides the analytical expression of return on equity. The realized monthly consumption growth and returns are aggregated to annual frequency, and they are compared to corresponding empirical moments.

In Table 2, the Dynamic Equity Premium column refers to the economy featuring stochastic volatility in growth. This economy is characterized by time-variation in both mean and volatility of growth, x_t and v_t. In contrast, the Constant Equity Premium column refers to the economy with constant volatility, where only the mean of growth is time-varying. In this economy, the expected growth in cash flow is state-dependent, but the equity premium is constant.

We set the consumption autocorrelation matrix A to be the same for these two economies but calibrate preference parameters differently so that each economy matches the empirical mean of asset returns. Thus, these two economies are almost the same concerning the autocorrelation and the standard deviation of consumption,[4] and the mean of eq-

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[4]Under the assumed structure of consumption growth, the standard deviation of consumption shock is theoretically the same irrespective of whether the volatility is stochastic or constant. This is because

$$\text{Var}\left(v_t^{1/2} e^T B W_{t+1}\right) = E\left[\text{Var}_t\left(v_t^{1/2} e^T B W_{t+1}\right)\right] + \text{Var}\left(E_t[v_t^{1/2} e^T B W_{t+1}]\right) = E\left[v_t e^T B B^T e\right] = v.$$

uity risk premium and the risk-free rate. The key differences between
the two economies are the presence of the equity premium dynamics and
the magnitude of kurtosis of consumption, as reported in the table. Both
economies undermatch the standard deviation of equity returns and over-
match the price-dividend ratio. However, this result is not surprising be-
cause the consumption stream that we calibrate here is not the leveraged
claim that Bansal and Yaron (2004) models.

The second step in the calibration is to specify project characteris-
tics – project lifetime N, size of cash flow d^0, and the maximum system-
atic risk $\beta_{\max}$ that characterizes the distribution of project-specific system-
atic risks. In calibrating these parameters, we use data on manufactur-
ing firms (SIC 2000-3999) from COMPUSTAT covering the years 1972
through 2017. We first pin down the project lifetime using the average
depreciation rate, Depreciation and Amortization (DPQ)/Property, Plant
and Equipment (PPEGTQ). Under the assumption that assets depreciate
exponentially at this average rate, the project lifetime is defined as the
time elapsed from the inception of a project until only 5% of the asset re-
mains. The empirical average of the depreciation rate is 0.288, implying
that projects last for approximately 10 years on average.

Next, we choose the maximum risk β_{max} of projects in the uniform
distribution to match the covariance between consumption and aggregate
income of the firms. We compute the quarterly aggregate income by sum-
ming individual manufacturing firms' observations of Operating Income
Before Depreciation (OIBDPQ) in each quarter. Then we compute the
ratio of covariance between consumption growth and operating income
growth to the variance of consumption growth. Note this is similar to how
we measure an asset's systematic risk in the CAPM. The resulting ratio is
3.68. To match this empirical ratio, we choose $\beta_{max} = 7.36$.[5]

Finally, we calibrate the projects' first cash flow d^0 to generate a real-

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However, the fourth central moment changes depending on whether the volatility is
stochastic or constant. Specifically, the fourth moment for the stochastic volatility is

$$E\left[\left(v_t^{1/2}e^T BW_{t+1} - E\left[v_t^{1/2}e^T BW_{t+1}\right]\right)^4\right] = v^2 + \frac{\sigma_w^2}{1-\varphi^2},$$

whereas it is v^2 for the constant volatility. Later, we consider an alternative calibration of
Type 1 pricing kernel to match the fourth moment to Type 2.

[5]The ratio of covariance between the consumption and the operating income to the
variance of the consumption is tightly linked to β_{max}. To illustrate, let d_t^i denote the project
i's cash flow at t, g_t the mean consumption growth, and β_i the systematic risk. Then, the

istic book-to-market ratio. The empirical average of the ratio is 0.669. To make our model reproduce this empirical moment, we conduct a simulation as follows. The simulation begins with 500 firms with no assets-in-place. Afterward, each month, the firms face a realization of the economic state that is common to all firms and firm-specific new projects. After observing the realized state and the project's systematic risk, each firm makes an investment decision. As time passes, each builds its own assets-in-place as a result of past investment decisions, while old projects expire. In this way, we generate a 20,000-month long firm panel. In this simulated panel, the book-to-market ratio is defined as the fraction of assets in place in firm value. The simulated average of this ratio is 0.631 with the choice of d^0 = 0.00901.

This calibrated model generates a pro-cyclical corporate investment, consistent with the finding in the business-cycle literature such as King and Rebelo (1999). Quantitatively, the correlation coefficient between consumption growth and investment growth is 0.40 in the simulated sample. This estimate is close to the empirical counterpart of 0.38.

4.5 Project Values and Firm-level Investment

A project is a collection of risky cash flows. To understand how the project value and investment rule vary across the economic states, we start by examining the state-dependence of the prices of the individual cash flows (the elementary assets). Figure 1 plots the prices of the elementary assets

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log of sum of cash flows from N projects is

$$\log(d_{t+1}^1 \cdots + d_{t+1}^N) = \log(d_t^1 e^{(g_t + \beta_1 v_t^{1/2} e^T B w_{t+1} - \beta_1^2 v_t e^T BB^T e/2)} \cdots + d_t^N e^{(g_t + \beta_N v_t^{1/2} e^T B w_{t+1} - \beta_N^2 v_t e^T BB^T e/2)})$$

$$\approx \log(d_t^1 e^{(g_t + \overline{\beta} v_t^{1/2} e^T B w_{t+1} - \overline{\beta}^2 v_t e^T BB^T e/2)} \cdots + d_t^N e^{(g_t + \overline{\beta} v_t^{1/2} e^T B w_{t+1} - \overline{\beta}^2 v_t e^T BB^T e/2)})$$

$$+ \frac{d_t^1 (v_t^{1/2} e^T B w_{t+1} - 2\overline{\beta} v_t e^T BB^T e/2)}{d_t^1 \cdots + d_t^N} (\beta_1 - \overline{\beta}) + \cdots + \frac{d_t^N (v_t^{1/2} B w_{t+1} - 2\overline{\beta} v_t e^T BB^T e/2)}{d_t^1 \cdots + d_t^N} (\beta_N - \overline{\beta})$$

where the Taylor expansion with respect to β_i is used around the weighted average of β_is, $\overline{\beta} = \frac{d_t^1 \beta_1 \cdots + d_t^N \beta_N}{d_t^1 \cdots + d_t^N}$. From the definition of $\overline{\beta}$, the first order terms in the expansion cancel out, leading to

$$\log\left(\frac{d_{t+1}^1 \cdots + d_{t+1}^N}{d_t^1 \cdots + d_t^N}\right) = g_t - \frac{\overline{\beta}^2 v_t e^T BB^T e}{2} + \overline{\beta} v_t^{1/2} e^T B w_{t+1}.$$

Thus, the unconditional covariance between consumption growth and aggregate income growth is $\overline{\beta} v$, which is $\overline{\beta}$ times the variance of consumption growth.

against their maturity in the economy with a dynamic equity premium. Generally, the price decreases with the asset maturity – the date at which the asset pays. Besides, the price also changes in response to a shift in the economic state – the conditional mean and the volatility of the consumption growth. As an illustration, the figure plots the asset prices at high and low state for each state variable by one standard deviation. Lastly, the price also depends on the project-level systematic risk β_t. The figure shows the asset prices when $\beta_t = 1$ in panel (a) and 3 in panel (b).

The price dependence on the expected growth, x_t, is straightforward. If the economy is expected to grow at a higher rate, the assets are expected to deliver a larger cash flow. Therefore, the asset price increases with the expected growth.

The price dependence on the conditional volatility, v_t, is two-fold. A rise in the volatility amplifies a negative covariance between the pricing kernel and the asset payoff, increasing the systematic risk. At the same time, the more volatile consumption growth reduces the certainty equivalent of future consumption. This makes the agent value future payoff more in general, thus resulting in the decline in the risk-free return. The trade-off between these two opposing forces depends on the delivery date of the assets. For short-term assets, the second effect dominates, so the price increases with the volatility. On the contrary, the price of long-term assets is more influenced by the first effect, resulting in a lower value in a state of higher volatility. The trade-off also depends on the project-level parameter β_t. For example, when β_t is as high as 3 in the panel (b) of the figure, the covariance channel outweighs the certainty-equivalent channel for all maturities of the assets. Hence, higher volatility reduces the asset prices across all maturities.

Figure 3 characterizes firm's optimal investment policy. The panel (a) describes the investment for the pricing kernel with a dynamic equity premium. Recall that the investment would occur on date t only when the project-level systematic risk β_t is below the threshold. The figure shows the likelihood of the investment conditional on the economic state x_t and v_t; $\text{Prob}(\beta_t \leq \overline{\beta}(x_t, v_t))$. Notice the investment differs across economic conditions. For example, as you would expect, the firm is more likely to invest in a state with high expected growth. This replicates the pro-cyclical behavior of aggregate investment, a stylized fact in business cycle literature like King and Rebelo (1999).

We are mainly interested in how investment rule should respond to the

volatility. The figure shows that the firm is less likely to invest when it faces a higher uncertainty. Investing less when volatility is high is a familiar result in the real-option literature, as in Dixit and Pindyck (1993). There a rise in volatility decreases investment because the uncertainty increases the value of waiting. In our setting, however, the investment is a now-or-never option, so there is no value in waiting. Instead, the dependence on volatility comes from its impact on the pricing kernel. In states of higher volatility, the equity premium is higher, leading to a decrease in the present value of future cash flows from a new project. As a result, firms invest less. The panel (b) of the figure shows the investment rule for the economy with static equity premium. The dependence on the expected growth rate, x_t, is similar. Since the equity premium and growth-volatility are constant, the investment rule only depends on the mean growth.

5 Dynamic Equity Premium and Value

Figure 3 illustrates how investment policies differ between the two pricing kernels. Does this difference matter? To answer, we consider two types of firms implementing these investment policies. A Type 1 firm acts as if the equity premium is constant. A Type 2 firm acts as if the equity premium is dynamic. We calculate the value of each firm in each of the two calibrated economies. Of the four combinations, we focus on the value implication of the two mismatches between firm type and economy (a Type 1 firm in the dynamic-risk-premium economy; a Type 2 firm in the constant-equity-premium economy) to measure the cost of a sub-optimal investment policy. The result, which we elaborate on below, is that the Type 1 firm's sub-optimality results in a loss of 11.7% of the firm value. In contrast, the Type 2 firm only experiences a 0.11% loss due to the mismatch.

Figure 4 highlights the (mis) matches between firm type and economy. The panel (a) is for the economy with a dynamic equity risk premium. In the panel, the solid line characterizes the optimal investment policy (a Type 2 firm) for this economy. Specifically, the policy is described as the threshold level of systematic risk ($\bar{\beta}$ in equation (12)); when the realized project-level systematic risk is below this threshold ($\beta_j \leq \bar{\beta}$), the investment occurs. Notice the optimal policy adjusts the threshold in response to a change in volatility, which is tightly linked to the equity premium. If the equity premium increases due to a rise in volatility, the firm lowers the

threshold, thereby tightening the investment policy. In contrast, the Type 1's policy, plotted in the dashed line, is insensitive to changes in volatility and leaves the threshold unchanged. This makes the investment policy sub-optimal. The panel (b) is an economy with a constant equity risk premium. Here the optimal policy is the solid and flat line of Type 1. The dashed line is the sub-optimal rule of Type 2.

Notice the two panels in Figure 4 appear symmetric. A firm with a sub-optimal investment rule sometimes over-invests and sometimes under-invests. However, over and under investment have different implications for cash flow timing, hence, different implications for valuation. Under-investment, here, means no investment so initial cash flows are zero (not negative) and future cash flows are also zero (not positive). The under-investment is sub-optimal since the value of the project is on net positive. Over-investment means an initial cash flow that is negative (not zero) and future cash flows that are positive (not zero). The key is that in an economy with a dynamic equity premium, Figure 4(a), the over-investment and the (sub-optimal) initial negative cash flow happens when the marginal value of a unit of consumption is particularly high (i.e., the high volatility state). In contrast, investing as if the equity premium were dynamic when in fact it is static leads to over and under investment behavior that is idiosyncratic – sub-optimal but not correlated with marginal utility.[6]

We measure the magnitude of these losses from sub-optimal investment by simulating our economies. On average, the Type 1 firm incurs a loss of 11.7% of the firm value. This quantity represents the value loss if a firm uses a constant discount rate inspired by the static CAPM when the actual equity premium is dynamic as macro-asset pricing literature finds. It is important to note that this sizable value loss is solely attributable to the failure to consider the time-variation in the equity premium; the Type 1 firm has otherwise correct understanding when it comes to the unconditional mean of the equity premium and the mean growth x_t. In contrast, the Type 2 firm in the economy with a constant equity premium realizes an average loss of only 0.1% of the firm value. These results quantify the ba-

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[6]The type 2 firm's investment policy conditions on the volatility level v_t. The valuation of the type 2 firm in the economy with a constant equity premium is based on simulating a process for v_t that the firm sees counter-factually. This needless variation in investment policy is costly to the type 2 firm. Notice, in contrast, if the stochastic volatility happens to be near the intersection of the two lines in Figure 4(b), $\sim 7.8 \times 10^{-5}$, the cost of the sub-optimality would coincidentally be zero.

sic message of Figure 4(a): over-investment when the equity risk premium is high is particularly costly.

In measuring the value loss, we have not explicitly considered firms' capital structure. In fact, as long as the economy is frictionless as in the Modigliani-Miller theorem, the firm value would be the same irrespective of firms' financing choices. What if we introduce certain friction so that capital structure should matter to firms' investment? We conjecture that the loss would be greater for a levered firm. Once a debt is issued, shareholders, who have control over investment decisions, receive residuals out of project cash flows after serving debtholders. Due to this priority rule, these residual claims fluctuate more than project cash flows as the economic condition switches. Therefore, cash flows to shareholders from new projects are more exposed to the systematic risk for a levered firm. In this case, we expect that the incorrect understanding of equity premium dynamics will lead shareholders to deviate more from the optimal investment rule. As a result, the sub-optimal policy would be even more costly.

As to our main question – whether the time-variation in equity premium matters to a firm's capital budgeting – our answer is yes. If the economy features the time-varying equity premium, whether or not to consider the variation results in a difference in firm values that is sizable: 11.7%.

5.1 Value Loss under Alternative Pricing Kernels

Obviously, the measurement of the value loss depends on how we model the pricing kernel. In the previous section, we tune the pricing kernel of the Type 1, which acts as if the equity premium is static, such that it still matches the average equity premium and the risk-free return to those of the Type 2. Although these asset returns play a crucial role in determining the discount rate, it is possible that firms may pay attention to moments other than the asset returns. For example, they may refer to an unconditional moment of consumption growth or the price-dividend ratio to discipline the valuation practice. If these moments are used as benchmark criteria, the Type 1 pricing kernel could be calibrated differently. Furthermore, it is also possible that a firm may implement an investment policy that is only partially sub-optimal; the firm recognizes changes in the equity premium but with an imperfect understanding of the dynamics. Lastly, we can consider alternative specifications of the correct pricing kernel to check the robustness of our main finding. In this section, we explore these

possibilities and examine the value implications under these alternative
settings.

5.1.1   Alternative Calibrations of the Type 1 Pricing Kernel

Here we consider three alternative moments that help us calibrate the Type
1 pricing kernel. They are the unconditional second moment of consump-
tion shock (we name this version of calibration Type 1.1), the uncondi-
tional fourth moment of consumption shock (Type 1.2), and the average
price-dividend ratio (Type 1.3). In all of Type 1.1 through 1.3, the volatil-
ity of consumption growth is constant over time, and so is the equity pre-
mium. Other parameters including the preferences and the consumption
autocorrelation A are identical to those of Type 2. Then, for each type, we
choose the volatility parameter v separately to match the target moment
(e.g, the price-dividend ratio for Type 1.3).[7]

    Table 3 shows the calibrated parameters and the value implications.
In an economy with a dynamic equity premium, following the Type 1.1
investment rule results in the value loss of 13.6% and the Type 1.2 re-
sults in the loss of 12.3%. Both estimates are slightly greater than the
Type 1's 11.7%. This implies that paying attention to the consumption
moments only rather than the average equity premium as in the Type 1
would exacerbate the sub-optimality of the investment rule. On the other
hand, using the price-dividend ratio helps to alleviate the sub-optimality;
the Type 1.3 investment rule incurs a 9.8% loss. In the calibration of the
Type 1.3, matching the price-dividend ratio leads to setting the volatility
level higher than the previous two calibrations. This high volatility induces
firms to engage less in the over-investment when the growth is highly un-
certain. The smaller loss for this particular investment rule reaffirms our
previous finding that the over-investment is particularly costly.

    One may consider adding these alternative moments as the calibration
criteria to further discipline the Type 1 pricing kernel that only matches
the average asset returns. Based on such consideration, we calibrate the
Type 1.4 and the Type 1.5. In the Type 1.4, the pricing kernel matches
the fourth moment of consumption shock as well as the averages of the

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[7]According to the model structure, setting the Type 1.1's volatility the same as the
mean volatility of the Type 2 leads to the same unconditional second moments between
the two types. Thus, the Type 1.1 calibration is what we would obtain by simply turning
off the stochastic volatility of the Type 2, keeping all other parameters unchanged.

equity premium and the risk-free return. In the Type 1.5, the pricing kernel matches the price-dividend ratio as well as these average assets.[8] In calibrating each of these types, we choose not only the volatility v but also preference parameters to match the multiple target moments. We find that following the investment rules based on these pricing kernels results in a smaller loss than the Type 1.1 through Type 1.3; the loss is 8.6% for Type 1.4 and 6.9% for Type 1.5. This finding supports the idea that that using more moments helps to better discipline the investment rule and to alleviate the sub-optimality of the Type 1. Nonetheless, the loss estimate is still discernible. Overall, these alternative calibrations result in a range of the loss estimates 6.9% - 13.6%, similar to the baseline result of 11.7%.

The last row of the table reports the value implications for the sub-optimally investing Type 2 in a constant-equity-premium economy where the correct pricing kernel is the Type 1.i ($i \in \{1, 2, 3, 4, 5\}$). The loss estimate is from 0.07% to 0.22%. Similar to the finding in the previous section, this loss is much smaller than the loss for the other mismatch in a dynamic-equity-premium economy. This is because the constant equity premium does not lead to a correlation between the pricing kernel and the sub-optimal investment.

The findings in this section let us draw a couple of practical implications in capital budgeting. The first and intuitive takeaway is that targeting more relevant moments to estimate the pricing kernel would help firms find the discount rate closer to the true value, even when firms have a wrong model of the discount rate. Second, if we are uncertain on whether or not the equity premium is dynamic, a robust policy would be to invest as if it is dynamic, considering that the Type 1 policy persistently incurs much larger loss than the Type 2 across different calibrations.

5.1.2   *Imperfect Knowledge of Equity Premium Dynamics*

So far we have studied the contrast of a firm that assumes the equity risk premium is constant when it is not. Next, we allow for firms that are

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[8]We do not report the calibration result where the average asset returns and the second moment of consumption shock are matched, because it is already done in the baseline calibration of Type 1. In calibrating the Type 1.5 pricing kernel, we could not find the set of parameters that lead to the perfect match for all of the three target moments; the price-dividend ratio of the Type 1.5 calibration is slightly off the benchmark value. This is because the asset returns and the price dividend ratio are tightly interconnected.

sub-optimal because they have an incorrect understanding of the equity premium dynamics. Specifically, we consider the persistence of the premium. Here, a sub-optimal investment rule recognizes time-variations in the equity premium, but the sub-optimality exists due to an incorrect estimate of the persistence of the equity premium. For different estimates of the persistence, we measure the value loss.[9]

In Figure 5, the panel (a) shows the value implications for incorrect estimates of persistence that range from 0 to 0.99. In the baseline calibration, the true level of persistence is 0.987. The sub-optimal rule incurs the largest loss, 13.6% of firm value, when the rule considers the equity premium having zero persistence. This zero persistence is equivalent to assuming a constant equity premium by the Type 1. As the level of persistence increases and becomes closer to the true level, the loss decreases gradually. Interestingly, we find that a small deviation from the true persistence leads to a sizable loss; for example, the persistence of 0.900 results in a 12.9% loss. This finding of the value loss that is highly sensitive to the persistence parameter underscores the importance of a precise understanding of the equity premium. When the equity premium dynamics are not taken into account to its full extent, as we empirically document in section 2, a firm incurs a substantial loss.

5.1.3 Alternative Calibrations of the True Pricing Kernel

Our pricing kernel follows the specification of the long-run risk economy. This setting facilitates our analysis, as it allows us to value the consumption assets and projects in the closed forms. However, as far as the dynamic equity premium is concerned, the long-run risk model is not the only framework capable of generating the dynamics. For example, the habit formation model by Campbell and Cochrane (1999) also predicts fluctuations in the equity premium. Then it is natural to conjecture that this alternative specification of the economy may find a different estimate of the value loss for the sub-optimal investment. In this section, we measure the value loss in this alternative specification of the true pricing kernel.

As an attempt to approximate the habit model within the framework of the long-run risk economy, we force the persistence of the equity premium

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[9]To differentiate the persistence, we change the autocorrelation φ of the consumption volatility in the model. Other parameters are set identical to those of the Type 2.

to be much lower than the baseline calibration. This is motivated by the fact that one of a few differences between these two models is the persistence of the equity premium. The equity premium is highly persistent in the long-run risk model, but not so much in the habit formation. Note that we here differentiate the true pricing kernel and the true persistence instead of the incorrect pricing kernel, which is discussed in section 5.1.2.

Figure 5(b) shows the value loss to the Type 1, which acts as if the equity premium is constant, in an economy with a dynamic equity premium. Thus, the sub-optimality is here due to complete ignorance of temporal changes in equity premium instead of a rather gradual discrepancy in the persistence of the panel (a). We let the true persistence of the equity premium vary from 0.01 to 0.99, having in mind that different levels of persistence represent different models of the true discount rate. To make a fair comparison across these models, we calibrate each to generate the same average of the equity premium despite the different levels of persistence. We do so by adjusting the volatility of volatility σ_w in equation (3) for each persistence φ.

The result is that, surprisingly, the loss is fairly stable as the persistence changes. When the autocorrelation of the equity premium is 0.99, the Type 1 incurs an 11.7% loss. As the autocorrelation decreases, the loss even increases slightly, and it becomes 13.7% of the firm value at the autocorrelation of 0.01. This result might seem counter-intuitive because one could expect the Type 1 pricing kernel to be closer to the correct model as the true persistence decreases. However, in order to generate the same the average equity premium despite the fall in persistence, the volatility of volatility should rise (see equation (A.18)). This also penalizes the sub-optimal rule significantly. As a result, the loss remains sizeable.

We interpret this finding as suggesting that our value implication is not limited to a particular model. Under alternative specifications of the correct pricing kernels, the value loss continues to be significant for the sub-optimal investment.

5.2 Industry-Specific Value Loss from Sub-Optimal Investment

We have examined the value implications for the representative firm. Next, we consider narrowing our focus to a specific industry. Because industries differ in investment characteristics and asset compositions, it is interesting to compare the cost of this sub-optimality across industries. To quantify

the value loss for each industry, we compute the industry-specific empirical moments regarding average asset maturity, book-to-market ratio, and the ratio of the covariance between operating income and aggregate consumption to the variance of consumption growth. We then choose the project parameters to reflect these industry-specific moments.

Table 4 presents the value loss for sub-industries of manufacturing sector. The estimates display notable differences across industries, ranging from 7.7% to 47.6%. As you might expect, the industries with longer asset maturities, such as Petroleum Refining and Paper tend to suffer a larger loss from the sub-optimal investment. This implies that using the correct model of pricing kernel is particularly important when evaluating longer-term projects. In addition, the industry's book-to-market ratio is correlated with the size of the value loss. An industry with a lower book-to-market ratio suffers more from the sub-optimal investment. As an example, let us compare Tobacco with Transportation Equipment. The two industries have the same average of asset maturity, but they differ in the book-to-market ratio with 0.475 for Tobacco and 0.695 for Transportation Equipment. We find that the difference leads to a much larger loss for Tobacco at 24.6%, compared to 9.2% for Transportation Equipment. This finding is intuitive in that having a correct investment rule is more important for the particular industries where growth options tend to constitute a larger part of the firm value.

5.3   *Firm Types and Returns*

We have shown that a firm with a sub-optimal policy that regards the equity premium as constant when it is in fact dynamic faces a large vale cost. And, in contrast, another mismatch – a policy that regards the equity premium as static when it is in fact dynamic – suffers only a small value loss. Next, we ask, how the sub-optimality impacts the expected returns. Firms that over- and under- invest will evolve to have different assets in place and the investment policy will impact the risk characteristics of the firm's growth options. Does this have a noticeable impact on the overall riskiness and expected returns?

We study the return implications by simulating the two economies. First, we consider an economy with a dynamic equity premium and simulate a history of 200,000 month-long economic states. In that setting, we place 500 pairs of firms. Each pair has a Type 1 firm (invests sub-

optimally; ignoring the dynamic equity premium) and a Type 2 (optimal investing in this economy) firm. Each pair sees idiosyncratic realizations of project-specific systematic risk, but two firms that belong to one pair see the same realizations. At the start all firms are identical with no projects in place. As time passes, each firm observes the relevant variables and invests according to its investment policy. This produces two groups – one consisting of Type 1 firms and the other consisting of Type 2 firms. Because both economy-wide and project-level shocks are controlled for, the difference in the firm assets between the two groups is attributable to their investment policies. In this simulated firm panel, we compute the returns on a portfolio consisting of Type 1 firms and and a portfolio of Type 2 firms. (We then repeat this exercise, analogously, for the economy with a constant equity premium.)

Table 5 reports the average return differences between the two firm types. The difference is shown for each economy in the top row of the table. Consistent with our finding on the valuation implications, investment policy has a larger economic impact in the economy with a dynamic equity premium. In an economy with dynamic equity premium, the monthly return on a Type 1 portfolio is on average 0.079% higher than that of Type 2 portfolio. The sub-optimally investing Type 1 firm is not only less valuable, it is riskier and generates a moderately higher expected return. In contrast, in an economy with a constant risk premium (here) sub-optimal Type 2 firm is riskier and has a higher expected return. However, in this economy the difference in returns across the two types is smaller than the difference in the other economy.[10]

Investment policy influences returns both through its impact on assets in place (past investments) and through growth options (future investments). Decompose the firm value, V_t into growth options, S_t, and assets in place, K_t. Then, total firm return, R_{t+1}, is :

$$
\begin{aligned}
R_{t+1} &= \frac{S_{t+1} + K_{t+1}}{V_t} \\
&= (1 - \mathrm{BM}_t)\frac{S_{t+1}}{S_t} + \mathrm{BM}_t\frac{K_{t+1}}{K_t} \\
&= (1 - \mathrm{BM}_t)R_{G,t+1} + \mathrm{BM}_t R_{A,t+1}
\end{aligned}
$$

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[10]The differences are statistically measurable, as seen in the shown t-statistics. However, this in part reflects that the level of firm-level volatility is low as well as being influenced by the arbitrary choice of having 500 firm portfolios.

where $BM_t = K_t/(S_t + K_t)$ is a book-to-market ratio, $R_{G,t+1}$ denotes return on growth options, and $R_{A,t+1}$ return on assets in place.[11]

The second and third rows of Table 5 show that in the economy with dynamic equity premium, the return difference between the two firms is driven entirely by the return to the growth-option component. The sub-optimal Type 1 and optimal Type 2 firms will have different assets in place, but across the two portfolio of firm types the Type 1's assets in place have similar risk and similar return (row 3). Recall from Figure 4, the optimally investing Type 2 firm sometimes accepts a high-beta risk project (when the equity premium is low) and sometimes rejects low-beta risk projects (when the equity risk premium is high). The sub-optimal Type 1 firm has a constant threshold. Quantitatively in our calibration, these policies produce Type 1 and Type 2 firms with assets-in-place with similar average beta-risk.

The difference in the returns of the two firm types comes from the riskiness of the growth options (row 2 of Table 5). The value of growth options is decreasing in the equity risk premium (returns are low to growth options when the risk premium goes up). These states are also where the Type 1 firm is most likely to over-invest and further reduce the value of growth options (by sub-optimally accepting a poor project). This extra loss co-occurs, as we saw earlier, with a higher marginal rate of substitution. Hence the extra risk, reflected in the expected rate of return, is associated with growth options of a sub-optimal investment in this economy.

6 Conclusion

In practice, capital investment decisions are typically hard business problems. They involve difficult and long-horizon forecasts, cut across many functional business areas, and are often strategic. The NPV rule and framework is a powerful tool for structuring this complex decision. The framework is, of course, not without many assumptions that do not strictly hold. In this paper, we look at one specific common practice in capital budgeting – discounting cash flows at a constant cost of capital. In our calibrated model, the implication is large. We estimate a 11.7% value loss from this decision. This loss is much larger than the 0.1% loss for the (counterfactual) scenario of a firm that is in a constant risk-premium economy but

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[11] Our model is agnostic about capital structure and dividend policy so we include $t+1$ cash flow from assets in place in K_{t+1} valuation.

invests according to a dynamic-risk-premium model. The dramatic difference is that the over-investment in the first case is correlated with bad states of the economy. We also provide empirical evidence that firms' response to time-variations in equity premium is statistically weak at the aggregate level. Still, it is quite possible that firms are heterogeneous in responding, so some firms do take into account the time-variations in the premium. One way we might infer which firms are investing with/without regard for the time-variation in the equity premium is through subsequent return behavior. In our calibrated model, monthly returns on the incorrectly discounting firms are 0.08% higher on average than those on correctly discounting firms. We leave an empirical exploration of this question to future research.

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Table 1: Empirical Relation Between the Investment-Implied Discount Rates and Stock Returns

Dependent Variable:					Realized Stock Returns					
	$\sum_{j=1}^{5} \rho^{j-1} \ln R_{t+j}$					$\sum_{j=1}^{10} \rho^{j-1} \ln R_{t+j}$				
Specifications:	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Implied Discount Rate$_t$	-0.005					-0.043				
	(-0.26)					(-1.58)				
Implied Discount Rate$_{t+1}$		0.011					-0.027			
		(0.53)					(-0.92)			
Implied Discount Rate$_{t+2}$			0.018					-0.013		
			(0.93)					(-0.45)		
Implied Discount Rate$_{t+3}$				0.020					-0.003	
				(1.06)					(-0.09)	
Implied Discount Rate$_{t+4}$					0.018					0.007
					(0.96)					(0.28)
adj-R^2(%)	-0.74	-0.56	-0.10	0.09	-0.11	2.05	0.82	-0.10	-0.57	-0.80
observations	128	128	128	128	127	123	123	123	123	123

The table shows the regression results where the realized quarterly returns on the portfolio of the manufacturing industry are regressed on the investment-implied discount rates. The dependent variable is the sum of the log returns for the next five years or the next ten years. The regression specification is

$$\sum_{j=1}^{N} \rho^{j-1} \ln R_{t+j} = \alpha + \beta \times (\text{Investment-Implied Discount Rate})_t + \epsilon_{t+N}$$

where N is either 20 (five years) or 40 (ten years) and the investment-implied discount rate is computed using equation (1). The t-statistics are based on the Newey-West standard errors and presented in parentheses below the parameter estimates.

Table 2: Calibration Results

		Data	Dynamic Equity Premium	Constant Equity Premium
State variables			x_t, v_t	x_t
Preference Parameters				
Risk aversion parameter	α		−9	−9.5
IES parameter	ρ		0.6	0.587
Endowment Parameters				
AR(1)	ϕ		0.92	0.92
MA(1)	θ		−0.75	−0.75
Volatility autocorrelation	φ		0.987	n.a.
Volatility of conditional variance	b		$2.3e^{-6}$	n.a.
Implications for Consumption Dynamics				
$AC(1)$		0.49	0.40	0.41
$AC(2)$		0.15	0.11	0.12
$AC(5)$		-0.08	-0.02	-0.02
$AC(10)$		0.05	-0.03	-0.03
$E[g]$ (%)		1.29	1.74	1.75
$\sigma(g)$ (%)		3.12	5.42	5.42
kurtosis(g)		1.69	2.97	2.91
Implications for Asset Returns				
$E[r_f]$ (%)		0.86	1.25	1.26
$\sigma(r_f)$ (%)		0.97	1.28	1.11
$E[r_e - r_f]$ (%)		6.33	5.26	5.25
$\sigma(E_t[r_e - r_f])$ (%)			1.04	0
$\sigma(r_e)$ (%)		19.42	7.16	6.94
Price-dividend ratio		35.69	59.30	58.13

The table shows the calibrated parameters and the resulting moments of consumption growth and asset returns. Data statistics are from Bansal and Yaron (2004). $AC(i)$ is ith autocorrelation of the yearly growth rate of consumption. r_e and r_f are yearly returns on equity and the risk-free bond, respectively. Other parameter values are $g = 0.0015$, $v = 0.008^2$, $\sigma_w = 0.23 \times 10^{-5}$, and $\kappa_1 = \beta = 0.997$.

Table 3: Value Loss under Alternative Calibrations

		Type 2 (benchmark)	Type 1.1	Type 1.2	Type 1.3	Type 1.4	Type 1.5
		Calibrated Parameters					
Volatility of consumption shock	v		$8.00e^{-3}$	$8.10e^{-3}$	$8.32e^{-3}$	$8.10e^{-3}$	$8.93e^{-3}$
Risk aversion parameter	α	-9	-9	-9	-9	-9.79	-8.59
IES parameter	ρ	0.6	0.6	0.6	0.6	0.542	0.474
		Moments to Compare					
Unconditional 2nd central moments of consumption shock		$6.40e^{-5}$	$\mathbf{6.40e^{-5}}$	$6.56e^{-5}$	$6.92e^{-5}$	$6.56e^{-5}$	$7.94e^{-5}$
Unconditional 4th central moments of consumption shock		$1.29e^{-8}$	$1.23e^{-8}$	$\mathbf{1.29e^{-8}}$	$1.44e^{-8}$	$\mathbf{1.29e^{-8}}$	$1.91e^{-8}$
Price-dividend ratio		59.30	63.48	62.16	**59.30**	54.30	**57.16**
$E[r_e - r_f]$ (%)		5.26	5.01	5.13	5.43	**5.26**	**5.26**
$E[r_f]$ (%)		1.25	1.37	1.30	1.13	**1.25**	**1.25**
		Value Implications					
Value loss to Type 1.i (%)			13.6	12.3	9.8	8.6	6.9
Value loss to Type 2 (%)			0.14	0.12	0.09	0.07	0.22

The table shows the alternative calibrations of the Type 1 pricing kernel and the resulting value loss for the sub-optimal investment. The Type 2 is the benchmark and the Type 1.i (i $\in \{12, 3, 4, 5\}$) is calibrated separately to match specific moments of the Type 2. The matched moments are presented in bold font. Type 1.1 is calibrated to match the unconditional second moment of consumption shock, Type 1.2 is to match the unconditional fourth moment of consumption shock, and Type 1.3 is to match the average price-dividend ratio. Type 1.4 is to match the averages of equity premium and the risk-free return in addition to the fourth moments of consumption shock. Type 1.5 is to match the averages of equity premium and the risk-free return in addition to the price-dividend ratio. In calibration of Type 1.1 through 1.3, the preference parameters and consumption autocorrelation parameter A are the same as those of Type 2. Only the consumption volatility parameter v is chosen for each type to match the target moment. In calibration of Type 1.4 and 1.5, preference parameters as well as v are chosen for each type to match the multiple moments. Value loss to Type 1.i is the loss in percentage of the firm value when Type 1.i investment rule is used in an economy with a dynamic equity premium. Value loss to Type 2 is the loss in percentage of firm value when Type 2 investment rule is used in an economy where the equity premium is constant and the correct pricing kernel is Type 1.i.

Table 4: Industry-Specific Value Loss from Sub-optimal Investment of Type 1

Industry	2-Digit SIC code	Asset Maturity (years)	Book-to-Market Ratio	Value Loss to Type 1
Food	20	16	0.634	47.6%
Tobacco	21	14	0.475	24.6%
Textile Mill	22	15	0.747	19.1%
Apparel and Fabrics	23	11	0.651	11.2%
Lumber and Wood	24	21	0.705	18.0%
Furniture and Fixture	25	16	0.671	10.8%
Paper	26	21	0.729	21.5%
Printing and Publishing	27	9	0.638	12.5%
Chemicals	28	8	0.454	12.3%
Petroleum Refining	29	22	0.723	33.0%
Rubber and Plastics	30	15	0.671	19.2%
Leather	31	11	0.639	8.8%
Stone, Clay, Glass, and Concrete	32	18	0.690	13.4%
Primary Metal	33	21	0.742	19.7%
Fabricated Metal	34	15	0.709	14.8%
Machinery and Computer Equipment	35	9	0.610	8.5%
Electronic and Electrical Equipment	36	9	0.592	8.5%
Transportation Equipment	37	14	0.695	9.2%
Measuring, Analyzing, and and Controlling Equipment	38	8	0.529	9.2%
Miscellaneous	39	9	0.648	7.7%

The table reports industry-specific estimates of value loss from Type 1 sub-optimal investment in an economy with a dynamic equity premium. For each industry, we choose the project parameters, N, d^0, and β_{max}, to match the empirical moments specific to the industry. The parameters for preferences and consumption dynamics are the same as the baseline calibration of the Type 2 pricing kernel.

Table 5: Return Differences between Type 1 and Type 2

Economy		Dynamic Equity Premium	Constant Equity Premium
r_t (type 1) $- r_t$ (type 2)	mean	0.079%	-0.011%
	t-stat	4.05	-22.93
$r_{G,t}$ (type 1) $- r_{G,t}$ (type 2)	mean	0.640%	0.0005%
	t-stat	34.03	1.87
$r_{A,t}$ (type 1) $- r_{A,t}$ (type 2)	mean	0.004%	-0.007%
	t-stat	1.54	-2.22

The table reports return differences between the two portfolios, one consisting of Type 1 firms and the other consisting of Type 2 firms in simulation of each underlying economy. r_t denotes the realized returns on firms, $r_{G,t}$ the returns on growth options, and $r_{A,t}$ the returns on assets in place.

Figure 1: Prices of Elementary Assets in the Economy with Dynamic Equity Premium

(a) Prices of Elementary Assets with $\beta = 1$
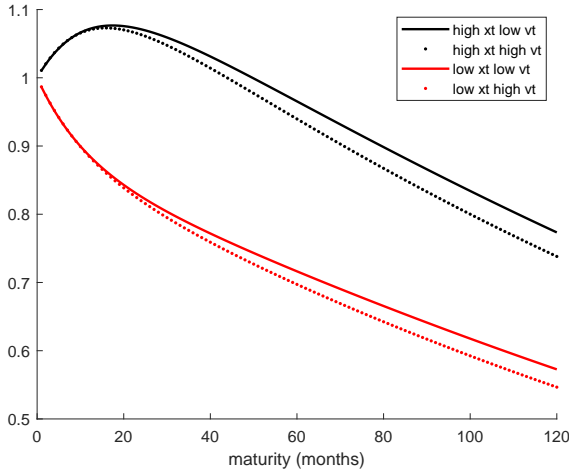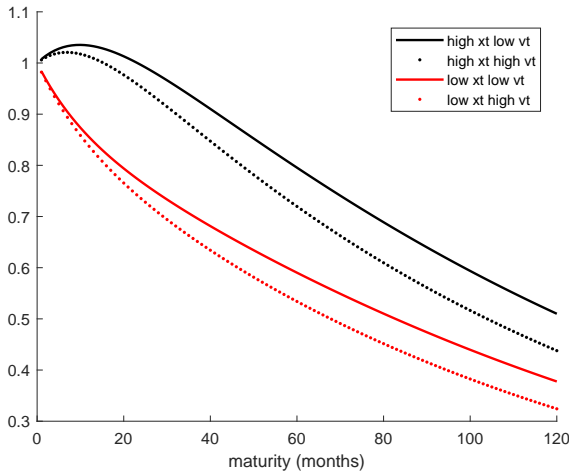

(b) Prices of Elementary Assets with $\beta = 3$

The figures plot the prices of the elementary assets in the economy with a dynamic equity premium. The panel (a) depicts the prices at different states of the economy against the delivery date, when the project-level systematic risk is $\beta = 1$. The panel (b) depicts the prices when $\beta = 3$.

Figure 2: Prices of Elementary Assets in the Economy with Constant Equity Premium

(a) Prices of Elementary Assets with $\beta = 1$
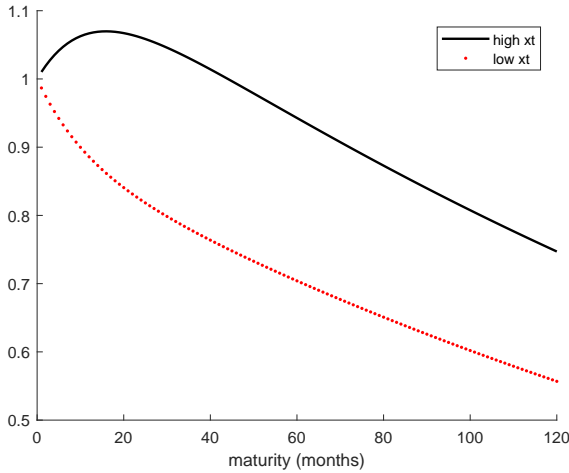
(b) Prices of Elementary Assets with $\beta = 3$
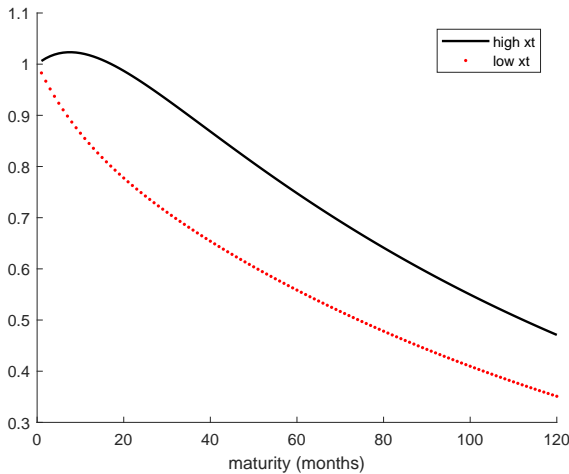
The figures plot the prices of the elementary assets in the economy featuring constant equity premium. The panel (a) depicts the prices at different states of the economy against delivery date, when the systematic risk is $\beta = 1$. The panel (b) depicts the prices when $\beta = 3$.

Figure 3: Characterizing Investment

(a) Dynamic Equity Premium
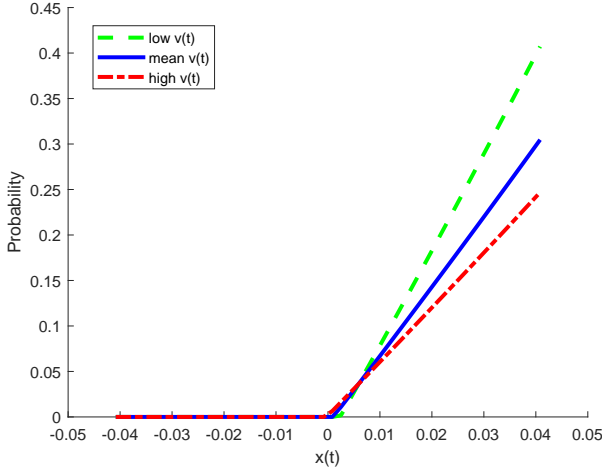
(b) Constant Equity Premium
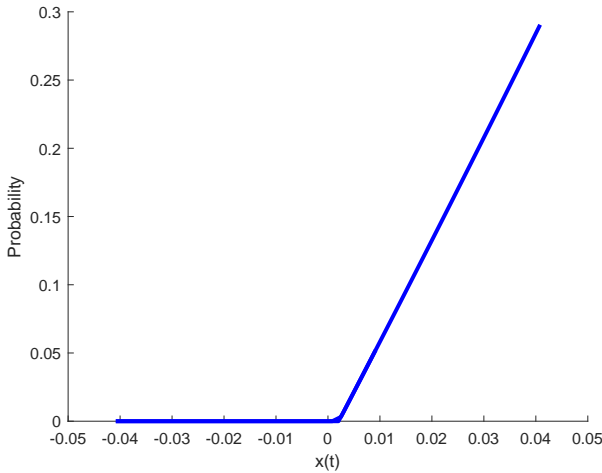
The figures plots the probability of investment conditional on the state of economy (x_t, v_t). Given the state variables, the investment occurs when the project-specific draw for systematic risk, β, is below the investment threshold, $\bar{\beta}(x_t, v_t)$. Then, the probability of investment is $\beta(x_t, v_t)/\beta_{max}$, as β is assumed to be drawn from a uniform distribution $[0, \beta_{max}]$.

Figure 4: Optimal and Sub-Optimal Investment Rules

(a) Economy with Dynamic Equity Premium
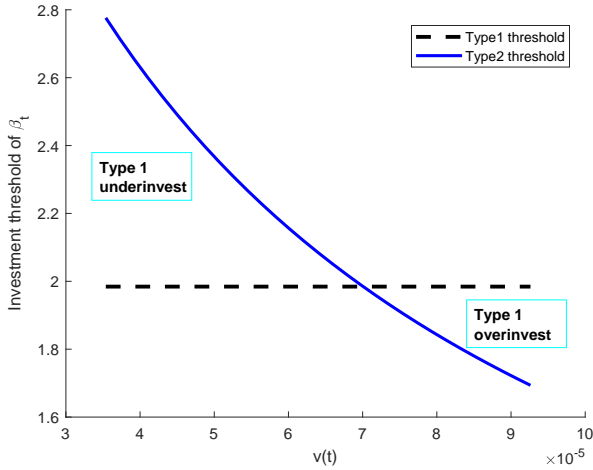
(b) Economy with Constant Equity Premium
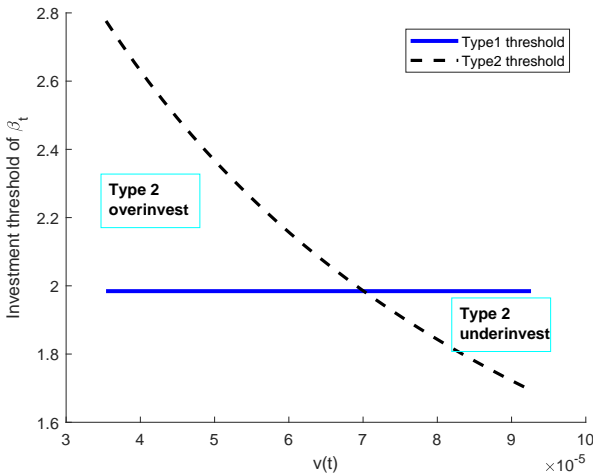

The figure plots the threshold of project-specific systematic risk $\bar{\beta}(x_t, v_t)$, below which the two firm types invest. Here x_t is set to be 0.0381 for an illustration. Panel (a) is for the economy with a dynamic equity premium. Panel (b) is for the economy with a constant equity premium. In each panel, the solid line depicts the threshold of the correct investment policy, while the dotted line depicts the threshold of the incorrect policy.

Figure 5: Persistence of Equity Premium and Value Loss

(a) Sub-Optimal Investment Rule with Incorrect Estimates of Persistence (true value $\varphi = 0.987$)
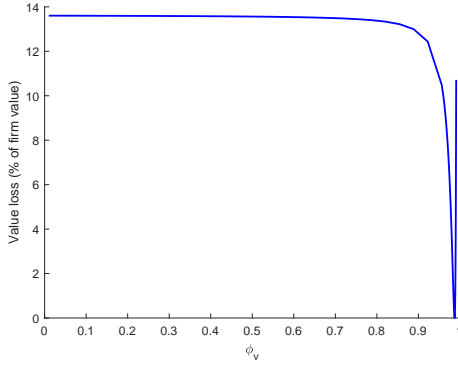
(b) Type 1 Sub-Optimal Investment Rule at Different Levels of True Persistence
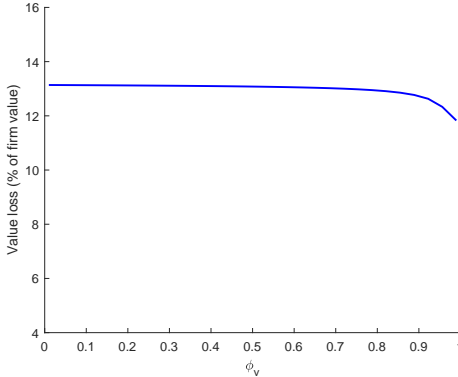
The panel (a) shows the value loss to a sub-optimally investing firm that has incorrect estimates of persistence of the equity premium, when the true persistence is 0.987 in an economy with a dynamic equity premium. Other parameters in the pricing kernel of the firm are identical to those of Type 2. The panel (b) shows the value loss to the Type 1 firm, which acts as if the equity premium is constant, at different levels of true persistence. For each level of persistence, the volatility of volatility σ_w is chosen so that the average equity premium is the same across different levels of persistence.

A Appendix

A.1 The Recovery of Investment-Implied Discount Rate

We analyze the empirical relationship between investments and discount rate at the portfolio level. For this analysis, we first compute the dividends and prices of the portfolio up to an arbitrary scale factor, using two returns on the portfolio, one inclusive and the other exclusive of dividends.

Next, we obtain individual firm's investment-capital ratio by dividing Capital Expenditures by the lagged Property, Plant and Equipment (Net) from COMPUSTAT. Then, to aggregate this ratio at the portfolio level, we calculate the value-weighted average of the investment-capital ratio, where the capital stock is the weight.

For date-t estimates of the dividend-capital ratio and future growth of dividends, we assume that firms have perfect foresight on these two quantities and use the realized values for the estimates. This is to prevent the possibility that an incorrect estimate of dividend growth may mask the investment-discount rate association. The analysis motivated by equation (1) requires the date-t estimate of the infinite sum of dividend growth. To calculate this, we assume that dividend will grow perpetually at a constant rate equal to the average of growth rates for the following five years since the date t. Finally, the parameter λ is set to be 4.35 following the calibration of Jermann (1998).

A.2 The Pricing Kernel

A.2 to A.5 are from Backus *et al.* (2010).

The pricing kernel in a representative agent model is the marginal rate of substitution between consumption at date t and consumption in state s at $t+1$. Define $\pi(s)$ as the probability of state s at $t+1$. Then the certainty equivalent is

$$\mu_t\left(U_{t+1}\right) = \left[\sum_s \pi(s)U_{t+1(s)}^{\alpha}\right]^{1/\alpha} \tag{A.1}$$

where $U_{t+1}(s)$ is continuation utility. Some derivatives of equation 2 and

equation A.1 are :

$$\frac{\partial U_t}{\partial c_t} = U_t^{1-\rho}(1-\eta)c_t^{\rho-1}$$

$$\frac{\partial U_t}{\partial \mu_t(U_{t+1})} = U_t^{1-\rho}\eta\mu_t(U_{t+1})^{\rho-1}$$

$$\frac{\partial \mu_t(U_{t+1})}{\partial U_{t+1}(s)} = \pi(s)U_{t+1}(s)^{\alpha-1}\mu_t(U_{t+1})^{1-\alpha}. \tag{A.2}$$

The marginal rate of the substitution between consumption at t and consumption in state s at $t+1$ is

$$\frac{\partial U_t/\partial c_{t+1}(s)}{\partial U_t/\partial c_t} = \frac{[\partial U_t/\partial \mu_t(U_{t+1})][\partial \mu_t(U_{t+1})/\partial U_{t+1}(s)][\partial U_{t+1}(s)/\partial c_{t+1}(s)]}{\partial U_t/\partial c_t}$$

$$= \pi(s)\eta\left(\frac{c_{t+1}(s)}{c_t}\right)^{\rho-1}\left(\frac{U_{t+1}(s)}{\mu_t(U_{t+1})}\right)^{\alpha-\rho}. \tag{A.3}$$

The rate of substitution without the probability is the pricing kernel used in the model.

A.3 Loglinear Approximation and Solution for the Scaled Utility

Note that the utility can be scaled by dividing current consumption with the use of homogeneity of both the time aggregator and the certainty equivalent function. If we define scaled utility $u_t = U_t/c_t$, then the equation can be scaled to

$$u_t = [(1-\eta) + \eta\mu(g_{t+1}u_{t+1})^\rho]^{1/\rho} \tag{A.4}$$

where $g_{t+1} = c_{t+1}/c_t$ is the growth rate of consumption. A first-order approximation of $\log u_t$ around $\log u$ is

$$\log u_t = \rho^{-1}\log[(1-\eta) + \eta\mu_t(g_{t+1}u_{t+1})^\rho]$$

$$= \rho^{-1}\log[(1-\eta) + \beta e^{\log\mu_t(g_{t+1}u_{t+1})^\rho}]$$

$$[(1-\eta) + \eta e^{\log\mu_t(g_{t+1}u_{t+1})^\rho}] \approx \kappa_0 + \kappa_1\log\mu_t(g_{t+1}u_{t+1}) \tag{A.5}$$

where

$$\kappa_1 = \frac{\eta e^{\log\mu}}{(1-\eta) + \beta e^{\rho\log\mu}}$$

$$\kappa_0 = \rho^{-1}\log[(1-\eta) + \beta e^{\rho\log\mu}] - \kappa_1\log\mu. \tag{A.6}$$

To get the solution for the scaled utility, guess the utility as function of x_t and v_t, in specific,

$$\log u_t = u + p_x^T x_t + p_v v_t. \tag{A.7}$$

Then, verify the function by plug the form into equation A.5 and compute for the coefficients $\{u, p_x, p_v\}$. First, compute the certainty equivalent:

$$
\begin{aligned}
\mu_t(g_{t+1} u_{t+1})^\alpha &= E_t\left[(g_{t+1} u_{t+1})^\alpha\right] \\
&= E_t\left[e^{\alpha(g + e^T x_{t+1} + u + p_x^T x_{t+1} + p_v v_{t+1})}\right] \\
&= E_t\left[e^{\alpha\left(g + u + (e^T + p_x^T)(A x_t + v_t^{1/2} B w_{t+1}) + p_v((1-\varphi)v + \varphi v_t + b w_{t+1})\right)}\right] \\
&= e^{\alpha\left(g + u + p_v(1-\varphi)v + \frac{1}{2} a p_v^2 bb^T\right) + \alpha(e^T + p_x^T) A x_t + \alpha\left(p_v \varphi + \frac{1}{2}\alpha(e^T + p_x^T) BB^T (e + p_x)\right) v_t}.
\end{aligned}
\tag{A.8}
$$

Here I used $Bb^T = 0$, and $E[e^x] = e^{a + \frac{1}{2}b}$ for $x \sim N(a, b)$. Plugging this to equation A.5 leads to

$$
\begin{aligned}
u + p_x^T x_t + p_v v_t &= \kappa_0 + \kappa_1\left[g + u + p_v(1-\varphi)v + \frac{1}{2} a p_v^2 bb^T + (e^T + p_x^T) A x_t \right. \\
&\quad \left. + \left(p_v \varphi + \frac{1}{2}\alpha(e^T + p_x^T) BB^T (e + p_x)\right) v_t\right]
\end{aligned}
\tag{A.9}
$$

The coefficients can be solved for as follows:

$$
\begin{aligned}
u &= \kappa_0 + \kappa_1\left[u + g + p_v(1-\varphi)v + \frac{\alpha}{2} p_v^2 bb^T\right] \\
p_x^T &= e^T(\kappa_1 A)(I - \kappa_1 A)^{-1} \\
p_v &= \frac{\alpha}{2}\kappa_1(1 - \kappa_1 \varphi)^{-1}(e + p_x)^T BB^T (e + p_x).
\end{aligned}
\tag{A.10}
$$

A.4 Derivation of the Pricing Kernel

We can substitute the scaled utility into equation 3. The pricing kernel has the term

$$
\begin{aligned}
\log(g_{t+1} u_{t+1}) - \log\mu_t(g_{t+1} u_{t+1}) &= g + e^T x_{t+1} + u + p_x^T x_{t+1} + p_v v_{t+1} \\
&\quad - \left(g + u + p_v(1-\varphi)v + \frac{1}{2} a p_v^2 bb^T\right) - (e^T + p_x^T) A x_t \\
&\quad - \left(p_v \varphi + \frac{1}{2}\alpha(e^T + p_x^T) BB^T (e + p_x)\right) v_t \\
&= v_t^{1/2}(e_t^T + p_x^T) B w_{t+1} + p_v^T b w_{t+1} \\
&\quad - \frac{\alpha}{2} p_v^2 bb^T - \frac{\alpha}{2}(e^T + p_x^T) BB^T (e + p_x) v_t
\end{aligned}
\tag{A.11}
$$

The pricing kernel follows as

$$
\begin{aligned}
\log m_{t+1} &= \log\eta + (\rho - 1)\log g_{t+1} + (\alpha - \rho)[\log(g_{t+1} u_{t+1}) - \log\mu_t(g_{t+1} u_{t+1})] \\
&= \log\eta + (\rho - 1)g - (\alpha - \rho)(\alpha/2) p_v^2 bb^T \\
&\quad + (\rho - 1)e^T A x_t - [(\alpha - \rho)(\alpha/2)(e + p_x)^T BB^T (e + p_x)] v_t \\
&\quad + v_t^{1/2}[(\rho - 1)e + (\alpha - \rho)(e + p_x)]^T B w_{t+1} + (\alpha - \rho)p_v w_{t+1} \\
&\equiv \delta_0 + \delta_x^T x_t + \delta_v v_t + v_t^{1/2}\lambda_x^T w_{t+1} + \lambda_v^T w_{t+1}
\end{aligned}
\tag{A.12}
$$

A.5 Equity Returns

We define equity as the consumption stream. The return is the ratio of its value at $t+1$, measured in units of $t+1$ consumption, to the value at t, measured in units of t consumption. The value at $t+1$ is U_{t+1} expressed in c_{t+1} units:

$$
\begin{aligned}
U_{t+1}/(\partial U_{t+1}/\partial c_{t+1}) &= U_{t+1}/\left[(1-\eta)U_{t+1}^{(}1-\rho)c_{t+1}^{\rho-1}\right] \\
&= (1-\eta)^{-1}u_{t+1}^{\rho}c_{t+1} \quad\quad\quad\quad (A.13)
\end{aligned}
$$

The value at t is the certainty equivalent expressed in c_t units:

$$
\begin{aligned}
q_t^c c_t &= \frac{\partial U_t/\partial \mu_t(U_{t+1})}{\partial U_t/\partial c_t}\mu_t(U_{t+1}) = \frac{\eta\mu_t(U_{t+1})^{\rho}}{(1-\eta)c_t^{\rho}}c_t \\
&= \eta(1-\eta)^{-1}\mu_t(g_{t+1}u_{t+1})^{\rho}c_t. \quad\quad (A.14)
\end{aligned}
$$

The return is the ratio:

$$
\begin{aligned}
r_{t+1}^c &= \eta^{-1}\left[u_{t+1}/\mu_{t+1}(g_{t+1}u_{t+1})\right]^{\rho}g_{t+1} \\
&= \eta^{-1}\left[g_{t+1}u_{t+1}/\mu_{t+1}(g_{t+1}u_{t+1})\right]^{\rho}g_{t+1}^{1-\rho} \quad\quad (A.15)
\end{aligned}
$$

The log of the return is

$$
\begin{aligned}
\log r_{t+1}^c &= -\log\eta + (1-\rho)g - (\rho\alpha/2)p_v bb^T \\
&\quad + (1-\rho)e^T A x_t - (\rho\alpha/2)(e+p_x)^T BB^T(e+p_x)v_t \\
&\quad + v_t^{1/2}(e+\rho p_x)^T Bw_{t+1} + \rho p_v bw_{t+1}. \quad\quad (A.16)
\end{aligned}
$$

The price of default-free bond which delivers 1 unit of consumption is $b_t^1 = E_t[M_{t+1}]$ and the return is $r_{t+1}^1 = \frac{1}{b_t^1}$. Thus the risk-free rate is

$$
\log r_{t+1}^1 = -(\delta_0 + \lambda_v^T\lambda_v/2) - \delta_x^T x_t - (\delta_v + \lambda_x^T\lambda_x/2)v_t. \quad\quad (A.17)
$$

Then, the excess return of equity is

$$
\begin{aligned}
\log r_{t+1}^c - \log r_{t+1}^1 &= (1/2)[(\alpha-\rho)^2 - \alpha^2]p_v^2 bb^T \\
&\quad + \left[\lambda_x^T\lambda_x/2 - (\alpha^2/2)(e+p_x)^T BB^T(e+p_x)\right]v_t \\
&\quad + v_t^{1/2}(e^T + \rho p_x^T)Bw_{t+1} + \rho p_v bw_{t+1} \quad\quad (A.18)
\end{aligned}
$$

A.6 Price of Elementary Assets

First, the date-t price-payout ratio of the asset that matures on the next date is:

$$
\begin{aligned}
q_t^1 &= E_t\left[m_{t+1}\frac{d_{t+1}}{d_t}\right]\\
&= E_t\left[e^{\delta_0+\delta_x^T x_t+\delta_v v_t+v_t^{1/2}\lambda_x^T w_{t+1}+\lambda_v^T w_{t+1}}e^{g+e^T(Ax_t+\beta_t v_t^{1/2}Bw_{t+1})-\frac{\beta_t^2 v_t}{2}e^T BB^T e}\right]\\
&= E_t\left[e^{\delta_0+\delta_x^T x_t+\delta_v v_t+g+e^T Ax_t-\frac{\beta_t^2 v_t}{2}e^T BB^T e}e^{v_t^{1/2}(\lambda_x^T+e^T B)w_{t+1}+\lambda_v^T w_{t+1}}\right]\\
&= E_t\left[e^{\delta_0+\delta_x^T x_t+\delta_v v_t+g+e^T Ax_t-\frac{\beta_t^2 v_t}{2}e^T BB^T e}E_{t+1}\left[e^{v_t^{1/2}(\lambda_x^T+e^T B)w_{t+1}+\lambda_v^T w_{t+1}}\right]\right]\\
&= E_t\left[e^{\delta_0+\delta_x^T x_t+\delta_v v_t+g+e^T Ax_t-\frac{\beta_t^2 v_t}{2}e^T BB^T e}e^{\frac{v_t}{2}(\lambda_x^T+e^T B)(\lambda_x^T+e^T B)^T+\frac{\lambda_v^T\lambda_v}{2}}\right]\\
&= e^{\delta_0+g+\frac{\lambda_v^T\lambda_v}{2}+(\delta_x^T+e^T A)x_t+\left(\delta_v+\frac{\lambda_x^T\lambda_x}{2}+\beta_t e^T B\lambda_x\right)v_t}\\
&\equiv e^{D_{0,1}+D_{x,1}x_t+D_{v,1}v_t} \qquad\qquad\qquad\qquad\qquad\qquad\text{(A.19)}
\end{aligned}
$$

Here we use the law of iterated expectation, and $Bb^T = 0$, $B\lambda_v = 0$, $b\lambda_x = 0$, and $E[e^x] = e^{a+\frac{1}{2}b}$ for $x \sim N(a,b)$.

Next, we compute the price-payout ratio of the elementary asset with maturity $s > 1$. Suppose that the price for the asset with maturity $s-1$ is given by

$$q_t^{s-1} = e^{D_{0,s-1}+D_{x,s-1}x_t+D_{v,s-1}v_t}. \qquad\qquad\qquad\text{(A.20)}$$

Then,

$$
\begin{aligned}
q_t^s &= E_t\left[m_{t,t+s}\frac{d_{t+s}}{d_t}\right] \\
&= E_t\left[m_{t,t+1}\frac{d_{t+1}}{d_t}m_{t+1,t+s}\frac{d_{t+s}}{d_{t+1}}\right] \\
&= E_t\left[m_{t,t+1}\frac{d_{t+1}}{d_t}E_{t+1}\left[m_{t+1,t+s}\frac{d_{t+s}}{d_{t+1}}\right]\right] \\
&= E_t\left[m_{t,t+1}\frac{d_{t+1}}{d_t}e^{D_{0,s-1}+D_{x,s-1}x_{t+1}+D_{v,s-1}v_{t+1}}\right] \\
&= E_t\Big[e^{\delta_0+\delta_x^T x_t+\delta_v v_t+v_t^{1/2}\lambda_x^T w_{t+1}+\lambda_v^T w_{t+1}+g+e^T(Ax_t+\beta_t v_t^{1/2}Bw_{t+1})-\frac{\beta_t^2 v_t}{2}e^T BB^T e} \\
&\qquad\times e^{D_{0,s-1}+D_{x,s-1}(Ax_t+v_t^{1/2}Bw_{t+1})+D_{v,s-1}((1-\varphi)v+\varphi v_t+bw_{t+1})}\Big] \\
&= e^{\delta_0+g+D_{0,s-1}+D_{v,s-1}(1-\varphi)v+\frac{(\lambda_v^T+D_{v,s-1}b)(\lambda_v^T+D_{v,s-1}b)^T}{2}+(\delta_x^T+e^T A+D_{x,s-1}A)x_t} \\
&\qquad\times e^{\left(\delta_v+D_{v,s-1}\phi_v-\frac{\beta_t^2}{2}e^T BB^T e+\frac{(\lambda_x^T+\beta_t e^T B+D_{x,s-1}B)(\lambda_x^T+\beta_t e^T B+D_{x,s-1}B)^T}{2}\right)v_t} \\
&\equiv e^{D_{0,s}+D_{x,s}x_t+D_{v,s}v_t}
\end{aligned}
\tag{A.21}
$$

In this way, we determine coefficients $\{D_{0,s}, D_{x,s}, D_{v,s}\}$ for any $s > 1$ recursively from $s = 1$.