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Cumulative Prospect Theory, Aggregation, and Pricing

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ABSTRACT

Cumulative Prospect Theory (CPT) has been used as a possible explanation of aggregate pricing anomalies like the equity premium puzzle. This paper shows that, unlike in expected utility models, a complete market is not sufficient to guarantee that the market portfolio is efficient and that the standard representative-agent analysis is valid. The separation or mutual fund theorems hold only under very restrictive conditions for CPT investors. Without them, aggregation breaks down, and assets are not necessarily priced as if there were one investor who behaved according to CPT. Under more limited conditions, the market portfolio can be efficient in a complete market with equally probable states. But in this case, individual CPT investors behave in the aggregate like a standard expected utility investor. Similarly, when faced with elliptically distributed assets, the capital asset pricing model (CAPM) holds for any combination of CPT investors and expected utility maximizers.

Keywords: Cumulative Prospect Theory, Two-Fund Separation, Optimal Portfolios, CAPM, Extreme-Risk Avoidance.

JEL Codes: G11, G12, C61

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Beginning in the 1970s, expected utility theory came under increasing question for failing to explain certain irregularities in behavior, and many modifications to the axioms or suggestions for alternate theories were proposed. Prospect Theory and, in particular, its successor, Cumulative Prospect Theory (CPT), is a response that has attracted a good deal of attention. As originally constructed by Kahneman and Tversky (1979) and extended by Tversky and Kahneman (1992), the theories have two complementary parts: S-shaped, loss-averse utility and a probability weighting function. Together these two features attempt a concise explanation of the major violations of expected utility theory and other seemingly incongruous behaviors or financial anomalies.

CPT has been tested in the laboratory where it has met with mixed results. For example, List (2004) says it only adequately explains the behavior of inexperienced agents. Those with more market experience "behave largely in accordance with neoclassical predictions." More importantly for finance, the applied literature on CPT has also been "tested" with attempts to explain pricing anomalies, such as the equity premium puzzle (Benartzi and Thaler, 1995), the size and value premium puzzles (De Giorgi *et al.*, 2004), and the disposition effect (Barberis and Xiong, 2009).

My paper takes a much more fundamental approach. It asks the questions: What kinds of portfolios would investors who are loss averse or use probability weighting assemble? How do these portfolios differ from those held by risk-averse investors or those who use objective probabilities? Do mutual fund theorems and their pricing implications continue to hold? In particular, is the market portfolio an efficient portfolio in the sense that a representative investor exists? And are assets then priced as if a representative agent is a CPT investor.[1]

CPT's non-concave utility can lead to extreme portfolio demands, and probability weighting is a form of heterogeneous beliefs, which can lead to different investors' optimal portfolios being quite dissimilar. My paper confirms that optimal portfolios for investors with S-shaped utility or who use probability weighting can differ substantially from optimal risk-averse

[1] Some of these questions have been previously addressed. In particular, De Giorgi *et al.* (2004) show some conditions under which a CAPM equilibrium does or does not obtain when investors have CPT preferences. Barberis and Huang (2008) ask a similar question with a somewhat conflicting answer. A CAPM equilibrium fails to obtain for heterogeneous TK-preferences agents (defined below in (1)) with $\alpha_i = \beta_i$, but does by default for homogeneous agents.

portfolios. In particular, demands across investors differ enough that even a complete market is insufficient to guarantee that the set of efficient portfolios is convex, as is guaranteed for risk-averse investors who use objective probabilities. This means that there may be no representative agent who holds the average or market portfolio. Unfortunately, a representative investor who optimally holds the market portfolio is, directly or indirectly, the basis for virtually all equilibrium models of asset pricing in Finance, and a representative agent is almost universally assumed when CPT pricing papers have calibrated their models to explain asset prices (e.g., Barberis and Huang, 2008).

This negative result is moderated to an extent because it is also shown that a representative investor does exist in a market with equally probable states (including markets with an atomless continuum of states). The resulting equilibrium is logically, though not necessarily statistically, equivalent to some classical equilibrium (one with only risk-averse investors utilizing objective probabilities). Unfortunately, this finding limits the types of anomalies that can be explained by CPT. When we can aggregate CPT investors into a representative agent, that representative agent is risk averse and uses the true probabilities.[2] In this sense CPT does not survive aggregation.

One might wonder how CPT performs in an incomplete market with limited trading opportunities. However, the context in which the CPT has been used to explain anomalies — the stock market — is unlikely to be a good example since derivative financial assets are so easily created. In any case, as aggregation is not an innocuous assumption under CPT, it is crucial to show exactly how the assumed representative investor behaves.

General two-fund separation results based either on utility (Cass and Stiglitz, 1970) or distributions (Ross, 1978) are quite limited under CPT. However, a more promising result is that the CAPM still obtains under portfolio weighting and, under mild conditions, for S-shaped and most other reasonable utility functions as well. Once again, though, the equilibrium is logically the same as for risk-averse investors who do not use probability weighting. The only difference between a mean-variance world in which investors are simply risk averse and one in which investors

[2]When investors use probability weighting, there may not be a unique representative agent; however, one of the possible representative agents will be risk averse and use the true probabilities.

are CPT investors is in the size of the market price of risk. This is, of course, not what earlier papers explaining anomalies with CPT have tested.

Section 1 provides a brief review of CPT and its two component parts, the S-shaped utility function and probability weighting. Section 2 introduces the portfolio maximization problem under CPT. Sections 3 and 4 examine the separate effects of S-shaped utility and probability weighting. Sections 5 and 6 scrutinize mutual fund theorems and mean variance analysis under CPT. Section 7 concludes.

1 Cumulative Prospect Theory: A Review

Prospect Theory (Kahneman and Tversky, 1979) and its successor, Cumulative Prospect Theory (Tversky and Kahneman, 1992), were introduced to explain violations of expected utility theory. Both theories share two components, an S-shaped rather than a concave utility function and a weighting scheme that differs systematically from the true probabilities.

For the standard expected utility problem, utility is strictly increasing and concave in wealth or consumption. Under CPT, utility is still strictly increasing but is reframed to be defined over the deviation, z, from some reference level; the function is normalized so that the utility of no change is zero, $v(0) = 0$. Rather than being concave, utility is S-shaped, concave above 0 and convex below 0. This means choices are risk averse concerning gains and risk seeking with regard to losses. This type of utility will be called S-utility for short. If utility is twice differentiable, except possibly at 0, then $v'(z) > 0$ for all z, and $z'' \cdot v(z) \leq 0$, $z \neq 0$. Though choices are risk seeking over losses, it is typically assumed that utility is loss averse. There are a number of different formal definitions of loss aversion in the literature.[3] The following definitions are used here.

[3]Tversky and Kahneman (1992) and Wakker and Tversky (1993) use the strong loss aversion definition, though others call this *increasing symmetric bet aversion*. Köbberling and Wakker (2005) define their measure of loss aversion as $\lambda \equiv v'(0^-)/v'(0^+)$, which is related to strong loss aversion but is directly applicable only at 0. Bowman *et al.* (1999) employ the strong definition but also consider the even more stringent condition $v'(x) \geq 2v'(y)$, $\forall x < 0, \forall y \geq 0$. Neilson (2002) uses the definition $v(x)/x \geq v(y)/y \forall x < 0\ \forall y > 0$. Note that all of the inequality definitions are properties of the utility function alone, independent of the probability weighting, even though the verbal definitions are about the preference relation as a whole, including the weighting function. A direct interpretation of the weak loss aversion verbal statement would be $-\Omega^-(1/2)v(-z) \geq \Omega^+(1/2)v(z)$ where

Definition. A utility function displays *weak loss aversion* if no symmetric fair binomial bet is preferred to the status quo; i.e., $v(z) + v(-z) \leq 0$, $\forall z > 0$. A utility function displays *strong loss aversion* if larger symmetric fair binomial bets are never preferred to smaller ones; i.e., $v(z_2)+v(-z_2) \leq v(z_1) + v(-z_1) \leq 0$, $\forall 0 < z_1 < z_2$. Either version of loss aversion is called strict if the relevant comparison is a strict inequality.

The latter definition is clearly the stronger one since it includes the former when $z_1 = 0$. If the gain-loss utility function is differentiable (except possibly at zero), strong loss aversion can be equivalently defined as $v'(-z) \geq v'(z)$, $\forall z > 0$.[4] For weak loss aversion, this marginal utility relation need hold only at zero; i.e., $v'(0^-) \geq v'(0^+)$. Both types of loss aversion are generalizations of risk aversion. For any standard utility of wealth function $u(W)$, define the gain-loss utility function $v(z) \equiv u(W_0 + z) - u(W_0)$. Then if v is loss averse, $[u(W_0 + z) + u(W_0 - z)]/2 \leq u(W_0)$, and u must be concave at every point W_0.

Tversky and Kahneman (1992, henceforth TK) proposed and estimated a specific S-shaped, loss-averse utility function of the form[5]

$$v(z) = \begin{cases} z^\alpha & z \geq 0 \\ -\lambda(-z)^\beta & z < 0 \end{cases} \quad \text{with } 0 < \alpha,\ \beta \leq 1 \text{ and } \lambda \geq 1. \tag{1}$$

Their estimated parameters are $\alpha = \beta = 0.88$ and $\lambda = 2.25$. Abdellaoui (2000) obtained the similar estimates $\alpha = 0.89$ and $\beta = 0.92$. Other estimates include $\alpha = 0.52$ and $\alpha = 0.37$ by Wu and Gonzalez (1996)

$\Omega^\pm$ are the probability weighting functions for gains or losses. This verbal preference-relation definition cannot be extended to symmetric fair bets with more than two outcomes without complete knowledge of the weighting functions.

[4]The derivative definition of strong loss aversion together with $v(0) = 0$ implies the levels definition by integration. The levels definition applied to x and $x + \varepsilon$ implies the marginal-utility definition provided v is differentiable. For weak loss aversion this relation holds at zero because $0 \geq v(\varepsilon)+v(-\varepsilon) = v(\varepsilon)-v(0)-[v(0)-v(-\varepsilon)] \to \varepsilon[v'(0^+)-v'(0^-)]$.

[5]Although the TK utility function permits distinct parameters, with $\alpha \neq \beta$, it displays weak loss aversion only if $\alpha = \beta$ as otherwise, the ratio $-v(-z)/v(z) = \lambda z^{\beta-\alpha}$ cannot exceed one for all positive z. This violation of loss aversion is often ignored as inconsequential. For $\alpha < \beta$, the largest symmetric bet that would be accepted has a size of $\lambda^{1/(\alpha-\beta)}$. This is less than 0.44 for the estimated TK parameter of $\lambda = 2.25$ regardless of α and β. If these small gambles are excluded (or the utility function is suitably modified for such gambles), then TK utility is strongly loss averse for all $\beta \geq \alpha$ and $\lambda \geq 1$ and strictly so if at least one inequality is strict.

using their own and Camerer and Ho's (1994) data.[6] These values of α would seem to indicate only very mild aversion to risk because the relative risk aversion for gains for this function is $1-\alpha$, and of course there is risk seeking for losses. However, the parameter λ also strongly affects the aversion to risk. For example, using TK's parameter values, a fifty-fifty gamble winning \$1 or \$2 has a certainty equivalent of \$1.49 — one cent below the expected value, but an even chance at winning or losing a dollar has a certainty equivalent of −23¢. So while risk aversion over gains is mild, there is much larger risk aversion when both gains and losses are involved.

Instead of using the outcomes' probabilities directly, CPT uses decision weights, ω, derived from a probability weighting function. Decision-weighted "expected" utility is computed as

$$\mathbb{E}_{\omega}[v(\tilde{z})] = \sum \omega_i(\boldsymbol{\pi}, \boldsymbol{z}) v(z_i). \tag{2}$$

Decision weights were originally introduced in Prospect Theory to capture two behavioral effects: (i) the subjective overweighting of rare events which seemed evident in behaviors such as the purchase of lottery tickets and (ii) violations of the independence axiom accounting for the Allais paradox.

As originally proposed, decisions weights could not readily be extended to gambles with more than two non-zero outcomes as violations of first-order stochastic dominance are introduced. Tversky and Kahneman (1992) developed CPT to surmount this problem. They accomplished this by applying weighting functions to the cumulative probability of losses and complementary cumulative probability of gains.

Under CPT, outcomes are first ordered from lowest to highest

$$z_{-n} < z_{-n+1} < \cdots < z_{-1} < z_0 = 0 < z_1 < \cdots z_m. \tag{3}$$

Then the cumulative probabilities for losses and the complementary cumulative probabilities for gains are determined

$$\Pi_{-i}^{-} = \pi_{-n} + \cdots + \pi_{-i} \qquad \Pi_i^{+} = \pi_i + \cdots + \pi_m. \tag{4}$$

Finally, weighting functions, $\Omega^{\pm}$, are applied separately to the cumulative and complementary cumulative probabilities, and the decision weights are

[6] Ho (1994) and Wu and Gonzalez (1996) consider only gains so the parameters β and λ were not estimated.

determined by differencing

$$\omega_{-i} = \Omega^{-}(\Pi^{-}_{-i}) - \Omega^{-}(\Pi^{-}_{-i-1}) \qquad \omega_i = \Omega^{+}(\Pi^{+}_{i}) - \Omega^{+}(\Pi^{+}_{i+1}) \quad \text{for } i > 0. \tag{5}$$

For a continuous distribution with an objective cumulative distribution $\Pi(s)$, the decision-weight density functions are

$$\omega^{-}(s) = \frac{d\Omega^{-}(\Pi(s))}{ds} = \frac{d\Omega^{-}}{d\Pi}\pi(s) \quad \omega^{+}(s) = -\frac{d\Omega^{+}(1-\Pi(s))}{ds} = \frac{d\Omega^{+}}{d\Pi}\pi(s) \tag{6}$$

where $\pi(s)$ is the objective probability density function over the states.

Impossible and certain events must be mapped in the obvious fashion so $\Omega^{\pm}(0) = 0$ and $\Omega^{\pm}(1) = 1$ for both weighting functions. In addition, each weighting function must be strictly increasing. If they are not, then a zero or negative decision weight can be assigned to a possible outcome. Furthermore, it seems desirable to make the weighting functions continuous so that a non-extreme event with a very small probability can never be assigned a large decision weight. In particular, a discontinuous weighting function can assign a decision weight atom to an atomless probability distribution. Finally Ω must also be differentiable if a smooth decision weight density function is to be achieved.

To explain the apparent overweighting of rare extreme events, the cumulative weighting functions have an inverted-S shape. The specific weighting functions proposed by Tversky and Kahneman are[7]

$$\Omega^{\pm}(\Pi) = \frac{\Pi^{\delta_{\pm}}}{[\Pi^{\delta_{\pm}} + (1-\Pi)^{\delta_{\pm}}]^{1/\delta_{\pm}}}, \tag{7}$$

and they estimated the parameters to be $\delta^{-} = 0.69$, $\delta^{+} = 0.61$. Other estimates for δ^{+} reported in Camerer and Ho (1994) range from 0.28 to 0.97, with one exception.[8] Prelec (1998) proposed the two-parameter

[7]The TK weighting function given in (7) is not monotonic for all parameter values; therefore, it can assign negative decision weights. For example, for $\delta = 0.25$, $\Omega(\Pi)$ is decreasing over the range of cumulative probabilities $1.56\% < \Pi < 23.62\%$. Negative decision weights would be a severe problem for the interpretation of CPT leading to inconsistencies such as the choice of first-order dominated payoffs. Fortunately, the TK weighting function is monotonic for all values of δ greater than the root of the equation $(1-\delta)^{2-\delta} = \delta^{1-2\delta}$. The critical root is $\delta \approx 0.279$, so the function is monotone in the empirically relevant range. See Ingersoll (2008).

[8]Sopher and Gigliotti (1993) reported $\delta = 1.87$. For values of δ greater than 1, the decision weights are smaller than the true probabilities for the rare extreme outcomes. Only gains were considered so δ^{-} was not estimated.

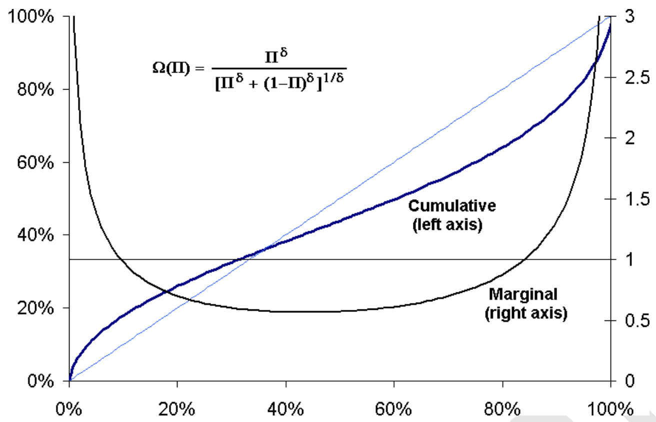

Figure 1: Tversky–Kahnemann probability-weighting function.

Description: This figure illustrates the inverted S-shaped probability-weighting function proposed by Tversky and Kahnemann, $\Omega(\Pi) = \Pi^{\delta}/[\Pi^{\delta} + (1-\Pi)^{\delta}]^{1/\delta}$ for the parameter $\delta = 0.65$.

Interpretation: The function over-weights the cumulative (complementary cumulative) probability where the curve is above (below) the 45-debree line. It over-weights the marginal probabilities in both tails where the marginal curve is greater than 1.

weighting function $\Omega^{\pm}(\Pi) = \exp[-\beta^{\pm}(-\ell n\Pi)^{\alpha}]$ based on an axiomatic derivation.

The TK weighting function is illustrated in Figure 1. For continuous distributions, the decision weight density for an objective probability density function of $\pi(s)$ is

$$\omega(s) = \Omega'(\Pi)\pi(s) = \frac{\delta\Omega(\Pi)}{\Pi}\left[1 - \frac{1-[(1-\Pi)/\Pi]^{\delta-1}}{\delta(1+[(1-\Pi)/\Pi]^{\delta})}\right]\pi(s). \tag{8}$$

For TK's parameters, $\delta_{-} = 0.69$ and $\delta_{+} = 0.61$, the cumulative probability of losses is over-weighted for probabilities less than 37.8% and the density is underweighted between the 12.4 and 82.2 percentiles; the gain complementary probability is over-weighted for probabilities less than 33.9% and the density is underweighted between the 10.9 and 82.8 complementary percentiles.

Note that the decision weight, ω_0, for an outcome of zero is unassigned by Equation (5). For CPT, the decision weight applied to a gain of zero might appear to be irrelevant because $v(0) = 0$ so ω_0 does not affect the computed "expected" utility. While this is true when CPT is used just to model choices, for equilibrium pricing results, it is the expectation of and covariance with marginal utility that matters so the numerical value of ω_0 is important in an equilibrium CPT model and needs to be assigned. Either Ω^- or Ω^+ could be extended to include a zero outcome; however, the two extensions do not in general give the same value for ω_0.

Another obvious completion is to set $\omega_0 = 1 - \Omega^-(\Pi_{-1}) - \Omega^+(\Pi_1)$, which makes the total of all the probability weights equal to one. Unfortunately, this will not always be practical. First there may be no outcome of zero to which to assign this weight. Second, this particular assignment will be negative in some cases unless *subcertainty* applies; i.e., $\Omega^-(\Pi) + \Omega^+(1-\Pi) \leq 1 \ \forall \Pi$.[9]

Even when the residual assignment for ω_0 is positive, it may differ markedly from the probability weights for similar outcomes, and pricing problems can arise. For example, TK's estimated decision weight functions with $\delta^- = 0.69$ and $\delta^+ = 0.61$ assign weights that total no more than 89% for all prospects whose zero outcomes occur at cumulative probabilities in the range 40% to 60%. Therefore, the no-change outcome must be assigned a probability weight of at least 11%. In many models with numerous outcomes, this could be an excessively large probability to assign to a single one. And if the underlying distribution is continuous, then an atom of decision weight at $z_0 = 0$ is required.

To avoid these problems and ensure that all probability weights are uniquely defined and positive, this paper makes the assumption that both the gain and loss weighting functions assign the same weight to the zero-gain outcome for all possible lotteries. This is equivalent to assuming a single weighting function for the cumulative distribution, ignoring the distinction between gains and losses.

Proposition 1 (Unique Zero-Gain Decision Weight Assignment). *The probability weighting functions, $\Omega^{\pm}$, both assign the same weight*

[9] For example, using TK's weighting function in (7) with $\delta_- = 0.9$ and $\delta_+ = 0.6$, a gamble with a 10% chance of a losses and an 85% chance of a gains has total probability weights of 18.80% and 82.25% for the losses and gains, which would require $\omega_0 = -1.05\%$. Subcertainty was assumed in the original version of Prospect Theory (Kahneman and Tversky, 1979) to eliminate preferences for some stochastically dominated gambles. This problem cannot arise in CPT where losses are assigned negative utility and weighting is applied to cumulative probabilities; nevertheless, it is generally assumed that subcertainty holds.

to the zero-change outcome for all gambles if and only if they satisfy $\Omega^-(\Pi)+\Omega^+(1-\Pi)=1\,\forall\Pi$. *This restriction is equivalent to using the single weighting function* $\Omega(\Pi)\equiv\Omega^-(\Pi)$ *on the entire cumulative distribution.*[10]

Proof. Denote by Π_{-1}, Π_1, and π_0, the cumulative probability of the smallest loss, the complementary cumulative probability of the smallest gain, and the probability of a gain of zero. The weight applied to the zero gain from extending the loss and gain weighting functions are $\omega_0^- = \Omega^-(\Pi_{-1}+\pi_0)-\Omega^-(\Pi_{-1})$ and $\omega_0^+ = \Omega^+(\Pi_1+\pi_0)-\Omega^+(\Pi_1)$. If $\omega_0^- = \omega_0^+$, then

$$\Omega^-(1-\Pi_1)+\Omega^+(\Pi_1)=\Omega^+(1-\Pi_{-1})+\Omega^-(\Pi_{-1}). \tag{9}$$

Because (9) applies to all risky prospects, Π_1 and Π_{-1} are arbitrary, and $\Omega^+(1-\Pi)+\Omega^-(\Pi)$ must be a constant for all Π. Furthermore, as $\Omega^\pm(0) = 0$ and $\Omega^\pm(1)=1$, this constant sum must be one. The converse is also obviously true.

Identify $\Omega(\Pi)\equiv\Omega^-(\Pi)$ as the single weighting function for the cumulative probabilities. This obviously assigns the same probability weights as Ω^- to all losses. Under the stated condition, it assigns the weight $\omega_s=\Omega^-(\Pi_s)-\Omega^-(\Pi_{s-1})=1-\Omega^+(1-\Pi_s)-1+\Omega^+(1-\Pi_{s-1})$ to any gain which is the same weight that Ω^+ assigns using the complementary cumulative probabilities. □

Throughout this paper, the maintained assumption is that all decisions are based on single, continuous, strictly increasing weighting functions, satisfying $\Omega(0)=0$ and $\Omega(1)=1$. From Figure 1, the effects of using one or two weighting functions are qualitatively similar in both tails so using a single weighting function will only alter the numerical results. This eliminates the difficulties discussed above and should not affect the qualitative properties of probability weighting.

A related advantage of using a single weighting function is that the resulting decision-weight distribution can be treated just like a subjective probability distribution. Standard methods and intuitions like stochastic dominance and Rothschild and Stiglitz (1970) riskiness can be applied directly to the decision weights. The one caveat is that these subjective

[10]Quiggin (1982), who first proposed the use of a cumulative weighting function, applied a single function to the entire distribution, though he imposed the additional condition that $\Omega(1/2)=1/2$. Analysis using a single weighting function is usually termed rank-dependent utility. De Giorgi *et al.* (2004) also use a single weighting function.

distributions belong to the risky prospects and not to any state space in which they are embedded. Two prospects with different orderings for their outcomes can have different subjective distributions even if they are defined on the same state space with given objective probabilities.

2 The Cumulative Prospect Theory Portfolio Problem

CPT was developed in the context of fixed gambles; that is, it was used to evaluate predetermined sets of outcome-probability pairs. However, to analyze portfolio problems, risky prospects whose outcomes are under some control of the decision maker must be compared. This leads to two distinct problems. First, in the standard portfolio problem unlimited buying and selling is allowed, and a convex valuation over losses may induce the investor to take unbounded positions. Second, the portfolio is chosen from amongst a set of assets with a known joint probability distribution, but the decision weights used in place of the probabilities cannot be determined until the ordering of the resulting portfolio outcomes across states is known. So as an investor evaluates different portfolios whose returns are not perfectly aligned, a changing set of decision weights may need to be employed in place of fixed state probabilities.

We will work in a standard single-period market setup. State s occurs with probability π_s and has a strictly positive state price of q_s.[11] Final wealth in state s is $W_s = I_0(1+\hat{x}+x_s)$ where I_0 is the total amount invested, $\hat{x}$ is the reference rate of return assigned zero utility, and x_s is the return in excess of $\hat{x}$, henceforth called the *subjective rate of return*.[12] Positive

[11]The assumption $q_s > 0$ assures there are no arbitrage opportunities. Although state prices can also be represented by risk-neutral probabilities, these "probabilities" cannot be subject to weighting because they are constrained by prices and possibilities not likelihoods. In particular, in a complete market, the risk-neutral probabilities are proportional to the state prices and are uniquely determined by the absence of arbitrage. Even in an incomplete market, the absence of arbitrage limits the feasible risk-neutral probabilities. These restrictions are identical across investors regardless of how they might weight probabilities.

[12]There is no loss of generality in defining utility in terms of subjective rates of return rather than dollar gains and losses in a single-period model. The reference level for dollar gains is $\hat{X} = I_0(1+\hat{x})$, and the dollar gain in excess of the reference level is $X_s = I_0(1+\hat{x}+x_s)-\hat{X} = I_0x_s$. It is only in a multi-period setting, in which wealth can change, that using rates of return rather than dollar gains and losses makes a difference due to reframing; see Ingersoll and Jin (2013).

subjective returns will be called gains, although this differs slightly from the objective meaning of the word that $\hat{x} + x > 0$. Negative subjective returns will be termed losses. Most commonly the reference level rate of return is set to either 0 or the interest rate, but any other value could be used.[13] The budget constraint is $\sum q_s(1 + \hat{x} + x_s)I_0 = I_0$ or,

$$\sum q_s x_s = 1 - (1 + \hat{x}) \sum q_s = (r_f - \hat{x})/(1 + r_f) \equiv B \tag{10}$$

where B has the same sign as $r_f - \hat{x}$, typically positive.

In performing the portfolio optimization, the decision weights, ω_s, are used in place of objective probabilities. The weights are affected by the ordering of the portfolio returns across states, but for a fixed ordering of returns, they are constant and can be used just like the state probabilities in the standard problem. Consider a particular ordering of portfolio outcomes across states, and with no loss of generality label the states so that $x_s \le x_s+1$.[14] The best portfolio with this specific ordering is the solution to

$$\text{Max} \sum \omega_s v(x_s) \quad \text{subject to} \sum q_s x_s = B \quad \text{and} \quad x_s \le x_{s+1}. \tag{11}$$

This proposed solution, however, guarantees only an order-constrained optimum. A different ordering of returns across states might have higher decision-weighted utility due to differences in the probability weights. Therefore, to solve the problem completely, the optimal portfolio for every ordering of returns must be determined, and their maximized decision-weighted utilities, $\sum \omega_s v(x_s^*)$, compared.[15] The optimal portfolio is the order-specific optimal portfolio that gives the highest expected utility.

The portfolio problem with no constraints apart from the budget and ordering constraints is a complete markets analysis. An incomplete-market portfolio problem can be analyzed by using additional constraints

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[13] If there is time-0 consumption, another natural choice is $\hat{x} = C_0(1+g)/I_0 - 1$ where g is a reference-level growth rate for consumption. The reference level can also be related to expectations (e.g., Kőszegi and Rabin, 2006).

[14] Typically under CPT all outcomes are distinct so if any two outcomes are equal, they should be merged into a combined state with a single probability weight. However, because states s and $s + 1$ are adjacent in the cumulative probabilities, the decision weight for the merged state will equal the sum of the two original decision weights; i.e., $\omega_{\{s,s+1\}} = \omega_s + \omega_{s+1}$. So when $x_s = x_{s+1} = x$, the contribution of these two states to expected (decision-weighted) utility can be expressed as either $\omega_s v(x) + \omega_{s+1} v(x)$ or $\omega_{\{s,s+1\}} v(x)$. Consequently, the weak inequality, which is usual for optimization, may be assumed.

[15] As a practical matter the constrained optimal portfolio need not be determined for all orderings. The constraint(s) that are binding in any one of the optimization problems will indicate which orderings to try. Propositions 4 and 5 restrict possible orderings.

restricting the feasible set of portfolio returns; i.e., $\mathbf{x} = \mathbf{X}\mathbf{y}$ where $\mathbf{X}$ is the $S \times N$ matrix of subjective returns on each of the N available assets in the S states, and $\mathbf{y}$ is a budget-constrained vector of the allocation to each of the N assets. Short sales restrictions or limited liability can be handled similarly by imposing $\mathbf{y} \geq \mathbf{0}$ or $\mathbf{x} \geq -(1+\hat{x})\mathbf{1}$, respectively.

The Lagrangian for (11) is $\mathscr{L} = \sum \omega_s v(x_s) + \eta[B - \sum q_s x_s)] + \sum_{s=0}^{S-1} \kappa_s (x_{s+1} - x_s)$. The first-order Kuhn-Tucker conditions are

$$\begin{gathered} 0 = \partial \mathscr{L} / \partial x_s = \omega_s v'(x_s) - \eta q_s - \kappa_s + \kappa_{s-1} \quad s = 1, \ldots, S, \\ 0 \leq \partial \mathscr{L} / \partial \kappa_s = x_{s+1} - x_s \quad 0 = \kappa_s (x_{s+1} - x_s) \quad s = 0, \ldots, S-1, \\ 0 = \partial \mathscr{L} / \partial \eta = B - \sum q_s x_s. \end{gathered} \tag{12}$$

The first line is standard apart from the constraint multipliers, κ_s. The second line imposes the ordering constraint.[16] When a constraint is not binding, the corresponding κ is zero, and that constraint does not affect the first-order condition in the top line. The final line is the budget constraint. Before proceeding with the analysis, two issues that do not arise in the standard problem must be mentioned. First, because utility is not concave, the maximization problem may not be well posed, and the optimal portfolio may not be bounded. Second, loss-averse utility functions typically have a kink at zero and are not differentiable there. This obviously can affect the first-order condition. Non-differentiability will be addressed in the next section. The boundedness of the optimal portfolio is discussed next.

To ensure that the optimal portfolio is bounded, a maximum tolerable loss is sometimes imposed. This worst allowed outcome is denoted by x_0; its value is fixed and not a choice variable unlike the other x's. For example, it might represent a total loss of wealth; i.e., $x_0 = -(1+\hat{x})$. Of course, a maximum loss can be imposed as a practical consideration even when the optimal solution would otherwise still be bounded below.[17] The maximum

[16]The constraint associated with κ_0 is that the smallest return, x_1, is not less than an exogenous lower bound x_0 (possibly $-\infty$). A restriction like this is required in some cases to ensure an optimal bounded portfolio and is discussed in more detail below.

[17]Assuming a maximum tolerable loss for an S-utility function essentially assigns a utility of $-\infty$ to any loss greater than x_0. This is similar to a Friedman and Savage (1948) utility function, which has lower and upper concave portions surrounding a convex portion. S-utility with a maximum tolerable loss specializes Friedman-Savage utility by making the lower concave portion infinitely risk averse and identifying the upper inflection point as the reference level. The maximum loss can also be made state dependent, and all

loss is set at $x_0 = -\infty$ to cover situations when no explicit maximum tolerable loss is to be imposed.

As a practical matter, there will always be some maximum tolerable loss (e.g., all wealth or the entire wealth of the economy) even if the value for x_0 cannot be precisely pinned down. However, in some models, it may be useful to avoid such an exogenous constraint. In such cases, bounded optimal portfolios can be assured by imposing some additional structure on the utility function. One assumption that ensures bounded portfolios is *extreme-risk avoidance*.

Definition 1. *A utility function displays extreme-risk avoidance (XRA) if*

$$\limsup_{x\to\infty} \frac{v(x)}{v(-kx)} = 0 \quad \forall k > 0. \tag{13}$$

Because utility is strictly increasing, $v(x) > 0$ for all positive x; therefore, it can display XRA only if v is unbounded for large losses. As portfolio formation allows only linear trade-offs between outcomes, XRA ensures that any "extreme" portfolio will be rejected as sub-optimal. In particular, a very large gain in some state must be financed by proportionally large losses in other states, but extreme-risk avoidance ensures that when the leverage is sufficiently large, the utility losses will more than offset the utility gains. This notion is made precise in Proposition 2.

Proposition 2 (Bounded Optimal Portfolios with Extreme-Risk Avoidance). *In a finite-state model, if an investor has extreme-risk avoidance and a zero-utility reference return less than or equal to the interest rate, then the optimal portfolio has bounded positions in every asset.*

Proof. The budget constraint puts an upper bound on the best return in terms of the worst return. By definition the smallest and largest subjective returns occur in states 1 and S, and from the budget constraint $B = \sum_{s=1}^{S} q_s x_s \geq x_1 \sum_{s=1}^{S-1} q_s + q_S x_S$ so $x_1 \leq Q(B - q_S x_S)$ where $Q^{-1} \equiv \sum_{s=1}^{S-1} q_s$. Because utility is increasing and the portfolio outcomes are

our results continue to hold, but there is little gain from this generalization and a large cost in complexity.

weakly ordered, the expected utility of this portfolio is

$$\begin{aligned}\mathbb{E}_\omega[v(x)] &= \sum_{s=1}^{S} \omega_s v(x_s) \le \omega_1 v(x_1) + v(x_S) \sum_{s=2}^{S} \omega_s \\ &\le \omega_1 v\left(Q(B - q_S x_S)\right) + v(x_S) \sum_{s=2}^{S} \omega_s. \end{aligned} \tag{14}$$

By the mean-value theorem, $v(Q(B - q_S x_S)) = v(-Q q_S x_S) + QBv'(aQB - Qq_S x_s)$ for some $a \in [0,1]$. If the utility function displays XRA, it is unboundedly negative as its argument becomes very negative, but v' is increasing for negative outcomes so as x_S grows large, $|v(Q(B - q_S x_S))| \sim |v(-Qq_S x_S)| \gg v(x_S)$, and the right-hand side of (14) will be negative.

Thus for any investor with XRA, portfolios with sufficiently extreme returns must have negative expected utility. Such a portfolio cannot be optimal if any portfolio with positive expected utility is possible, and that must be true if the zero-utility reference return is less than or equal to the interest rate because the risk-free portfolio then has nonnegative utility.[18] Therefore, all optimal portfolios must have bounded positions. □

If XRA is severely violated and the limit in (13) is infinite, for example under TK utility with $\alpha > \beta$, then similar reasoning shows that some unbounded portfolio is always optimal. On the other hand, if the limit in (13) is positive and finite, then the boundedness of optimal portfolios is indeterminate in general. To illustrate, consider a TK utility investor with $\alpha = \beta$ and a reference return $\hat{x} = 0$ in a two-asset, two-state economy with $q_1 = q_2 \equiv q < 1/2$ and $\omega_1 > \omega_2$. For all feasible portfolios $x_2 = B/q - x_1$, and expected utility for any portfolio with x_1 positive is[19]

$$\mathbb{E}_\omega[v(x)] = \begin{cases} \omega_1 x_1^\alpha - \omega_2 \lambda (x_1 - B/q)^\alpha & x_1 \ge B/q \\ \omega_1 x_1^\alpha + \omega_2 (B/q - x_1)^\alpha & 0 < x_1 < B/q. \end{cases} \tag{15}$$

[18]Even if the reference rate of return exceeds the risk-free rate, a negative expected utility portfolio cannot be optimal if the investor receives zero utility from not entering the market at all.

[19]Because the state prices are equal, the return in the more likely state must exceed that in the less likely state; furthermore, at least one return must be positive so only portfolios with $x_1 > 0$ can be optimal. If $\omega_2 < \omega_1 < \lambda\omega_2$, the optimal portfolio holds $x_1 = [1 + (\omega_1/\omega_2)^{1/(\alpha-1)}]^{-1} B/q < B/q$.

This can be increased without limit if $\omega_1 > \lambda\omega_2$ so the optimal portfolio is unbounded. On the other hand, if $\omega_1 < \lambda\omega_2$, the optimal portfolio will have both x_1 and x_2 positive and finite.

XRA is a different property from loss aversion though they are related. Loss aversion is neither necessary nor sufficient for XRA. The TK utility function in (1) has extreme-risk avoidance only if $\alpha < \beta$. As shown previously in footnote 5, it displays loss aversion only for $\alpha = \beta$. And as shown in the previous example, even strong loss aversion is insufficient to guarantee bounded optimal portfolios.

XRA is more closely related to He and Zhou's (2011) Large Loss Aversion Degree, defined as LLAD $\equiv \lim_{x\to\infty}[-v(-x)/v(x)]$. An investor displaying XRA will have LLAD $= \infty$. An infinite LLAD is required to ensure that an investor will have a finite demand for borrowing to lever a single risky asset or portfolio. XRA is a stronger condition ensuring that both unlimited leverage and infinite short positions in all risky assets are suboptimal.

In the remainder of the paper, every investor's optimal portfolio is assumed to have bounded outcomes due either to explicit bounds or XRA. The problem of characterizing optimal portfolios and answering the related question of how prices are set in a CPT equilibrium can now be addressed. The next section examines optimal portfolios when investors have S-shaped utility but use objective probabilities. Section 4 then addresses the effects of probability weighting.

3 The S-Utility Portfolio Problem

To focus on S-utility in this section, probability weighting is ignored and the objective probabilities are used. By standard reasoning, the absence of arbitrage guarantees that the realized rates of return r_s on any asset are related by $1 = \sum q_s(1+r_s) = \sum \pi_s\theta_s(1+r_s)$ where $\theta_s \equiv q_s/\pi_s$ is the state price per unit probability or stochastic discount factor when considered a random variable. The marginal utility of any strictly risk-averse investor can be used for θ, but under what conditions can marginal S-utility be used? And can an S-utility investor serve as the representative investor so that θ is the marginal utility provided by the market portfolio?

When solving the portfolio problem without probability weighting, the outcome ordering constraints are ignored and all those Lagrange multipliers are zero. For all states except possibly those with the maximum tolerable loss or a gain of zero, the first-order conditions in (12) hold as equalities and[20]

$$\begin{aligned} &v'(x_s) = \eta q_s/\pi_s && \text{for } x_s \neq x_0 \quad \text{and} \quad x_s \neq 0 \\ &v'(x_s) \leq \eta q_s/\pi_s && \text{for } x_s = x_0 \\ &v'(0^+) \equiv \lim_{x\downarrow 0} v'(x) \leq \eta q_s/\pi_s && \\ &\qquad\qquad \leq v'(0^-) \equiv \lim_{x\uparrow 0} v'(x) && \text{for } x_s = 0. \end{aligned} \tag{16}$$

The first line is the standard first-order equality condition, which is the same as that for a risk-averse investor. The second line applies to the states in which the maximum tolerable loss is realized; it is the same first-order condition but as an inequality because the investor might prefer to decrease x_s if unconstrained. If the utility function has a kink at zero with $v'(0^+) < v'(0^-)$, then the first-order condition for any state with a realized gain of zero is a two-sided inequality. For some S-shaped utility functions, including TK utility, marginal utility is unbounded near zero so their optimal portfolios will have no realized returns of zero, and, indeed, no returns that are close to zero (unless the ratio q_s/π_s is also unbounded) so the third line of (16) will be immaterial.

From (16) many of the properties of optimal portfolios are determined by the price-probability ratio, $\theta_s = q_s/\pi_s$, just as under global risk aversion. For states with an unconstrained return, $x_s^* = v'^{-1}(\eta\theta_s)$. Because $v''(x) < 0$ for risk-averse investors their optimal portfolios always have higher returns in states with smaller θ_s, and it is natural when comparing states to call the one with the lower price-probability ratio the better state. However, for S-utility investors, the states in which subjective returns are positive and negative must be examined separately. In addition, states where the maximum loss or a gain of zero are earned and the first-order

[20]The first-order conditions must hold even in the convex region of losses provided the loss is not maximal or zero. Consider any optimal portfolio and the same portfolio with two returns altered to $x_i \to x_i + q_j\varepsilon$ and $x_j \to x_j - q_i\varepsilon$. This alteration is affordable and the change in expected utility for small ε is $\Delta\mathbb{E}[v] = [\pi_i v'(x_i)q_j - \pi_j v'(x_j)q_i]\varepsilon + O(\varepsilon^2)$. If the first-order conditions do not hold, then the term in brackets is positive (negative). An increase (decrease) in x_i will then increase expected utility and the original portfolio could not have been optimal. This does not depend on whether utility is locally concave or convex just so long as neither x_i nor x_j is minimal or zero so either can be decreased as necessary. The example below shows that optimums with non-maximum tolerable losses do exist.

condition does not hold as an equality must also be considered. Proposition 3 gives an initial characterization of efficient S-utility portfolios.

Proposition 3 (S-Utility Optimal Portfolios in a Complete Market). *If the market is complete, the rates of return realized on the portfolio optimal for an S-utility investor with either XRA or a finite maximum tolerable loss* ($x_0 > -\infty$) *are characterized by:*

(i) Gains are larger in better states; i.e., for $x_i,\ x_j > 0$, $x_i > x_j$ *if and only if* $\theta_i < \theta_j$.

(ii) If there is no maximum tolerable loss, then a loss is realized in at most one state. If there is a maximum tolerable loss, then multiple states can suffer this loss, but there is only one or zero state with a loss not maximal in size.

(iii) Any state with a realized loss cannot have a smaller state price and the same or higher probability than any other state with a gain or a smaller absolute loss. However, portfolio outcomes need not be monotonic in θ_s.

Proof. All results follow immediately from the first-order conditions. The separate characterizations are verified and discussed below. □

Among states where gains are earned, the portfolio return will be higher the lower the price-probability ratio, just as for a risk-averse investor, because the first-order conditions hold and marginal utility is decreasing over gains. This verifies property (i). In some settings an S-utility investor's optimal portfolio may have gains in all states. This will occur, generically if the market does not provide sufficient reward for bearing risk.[21]

To verify the second property, suppose there are two states with non-maximum tolerable losses in the optimal portfolio. The same portfolio with just these two returns altered to $x_i \to x_i - q_j\varepsilon$ and $x_j \to x_j + q_i\varepsilon$ is affordable for any ε. A second-order Taylor expansion for the change in expected utility for this alteration is

$$\Delta\mathbb{E}[v(x)] = \varepsilon[\pi_j q_i v'(x_j) - \pi_i q_j v'(x_i)] + \frac{1}{2}\varepsilon^2[\pi_i v''(x_i) q_j^2 + \pi_j v''(x_j) q_i^2] + o(\varepsilon^2). \tag{17}$$

[21] For example, consider a two-state economy characterized by $\pi_1 = \pi_2 = 1/2, q_1 = 0.6$, $q_2 = 0.2$. An investor with the TK utility function with $\alpha = \beta = 1/2, \lambda = 2$, and $\hat{x} = 0$ optimally holds the portfolio $x_1 = 1/12$ and $x_2 = 3/4$.

The first term is 0 by the first-order conditions. The second term is positive because $v'' > 0$ in each state by assumption. Therefore, expected utility can be increased by this alteration, and the original allocation could not have been optimal. That is, a loss that is less than maximal can be realized in at most one state.[22] Of course, if there is no maximum tolerable loss ($x_0 = -\infty$), then it follows immediately that at most one state can have a loss verifying property (ii).

The intuition for this result is that investors with S-utility will always benefit by making a large loss worse in order to reduce a smaller loss if this is possible. With an ordinary risk-averse utility function, this is never beneficial as the increased loss has the bigger impact on utility. But in the convex portion of an S-utility function, it is better to concentrate losses in a single state because the marginal utility decreases rather than increases as the loss size is increased. Only if there is a maximum tolerable loss can an optimal portfolio realize losses in two or more states because the state price of a single state may not be large enough to finance the gains desired in all the other states.

To verify property (iii) consider two states with $\pi_i \le \pi_j$ and $q_j < q_i$. If some portfolio with $x_j = \ell \le 0$ and $x_i = h > \ell$ is affordable, then the portfolio with $x'_j = h + (q_i - q_j)(h - \ell)/q_j > h, x'_i = \ell$, and identical returns in all other states is also affordable and has higher expected utility because it first-order stochastically dominates the first portfolio.[23] Therefore, the original portfolio with the loss in the better state could not have been optimal. However, this property is not as strong as the reverse ordering between θ_s and x_s under risk aversion. For example, in the economy $\pi = (0.75, 0.25)'$, $\mathbf{q} = (0.8, 0.2)'$, the optimal portfolio for the TK utility function with $\alpha = 0.4$, $\beta = 0.6$, $\lambda = 2$, and $\hat{x} = 0$ is $\mathbf{x}^* = (0.0051, -0.0205)'$. The price-probability ratios are $\boldsymbol{\theta} = (1.07, 0.8)'$, so the optimal portfolio has its higher return in the worse state.

This property might seem to be only a curiosity, but it is more than that. When the returns on all optimal portfolios are not monotonically related, the set of efficient portfolios for risk-averse and S-utility investors need not be convex even in a complete market, as it always is among risk-averse

[22] Equation (17) cannot be applied if one of the states already has the maximum tolerable loss. Only alterations that increase a maximum tolerable loss are possible and the first-order condition may not hold as an equality.

[23] First-order stochastic dominance is a valid comparison for S-utility because the only requirement for first-order stochastic dominance is increasing utility.

investors. And when the efficient set is not convex, the market portfolio need not be efficient.[24]

To illustrate that the market portfolio need not be efficient, consider the three-state economy with $\pi = (0.35, 0.4, 0.25)'$ and $\mathbf{q} = (0.375, 0.425, 0.2)'$. A TK-utility investor with $\alpha = 0.4$, $\beta = 0.6$, $\lambda = 2.5$, and $\hat{x} = 0$ optimally holds the portfolio $\mathbf{x}' = (0.0051, 0.0052, -0.0206)$ whose loss is not in the worst state. This does not violate property (iii) because the third state has a smaller probability than the first. Combining this portfolio with one whose returns are ordered inversely to θ can lead to a market portfolio that is not efficient. For example, a TK-utility investor with $\alpha = 0.8$, $\beta = 0.9$, $\lambda = 2$, and $\hat{x} = 0$ optimally holds the portfolio $\mathbf{x}' = (-0.0037,\ 0.0011,\ 0.0045)$.[25] If one-quarter of the investors are of the first type and three-quarters are of the second type, then the aggregate demand of the market is $\mathbf{x}_{\mathrm{mkt}} = (-0.0015,\ 0.0021,\ -0.0017)$. This cannot be the optimal portfolio of any unconstrained S-utility investor because two subjective returns are negative, which is prohibited by property (ii). Nor can it be the optimal portfolio of any risk-averse investor because the returns are not ordered inversely to θ. Therefore, this economy can have no representative investor who is either of these types. Consequently pricing results that rely on an assumed optimal market portfolio will be invalid.[26] The existence of a representative investor and efficiency of the market portfolio are discussed in more detail in Section 5.

Figure 2 illustrates and Table 1 describes the optimal portfolios for S-utility investors with a one-year investment horizon in an economy with a lognormal market portfolio. The market portfolio is characterized by $r = 5\%$, $\mu = 13\%$, $\sigma = 20\%$ with state prices determined assuming a representative investor with a constant relative risk aversion of 2. Investors have TK utility with parameters $\lambda = 2.25$, $\alpha = \beta = 0.88$ or 0.5, a reference return of $\hat{x} = 0$, and maximum tolerable losses of $x_0 = -100\%$ or -25%. The optimal portfolios' rates of return are plotted against the rates of

[24]See Dybvig and Ross (1982) and Ingersoll (1987) for examples of the nonconvexity of the efficient set among risk-averse investors when the market is incomplete.

[25]This second portfolio has its returns ordered inversely to θ so it is the optimal portfolio for some strictly risk-averse investors as well. Therefore, the inefficiency of the market portfolio does not require heterogeneous CPT investors.

[26]It might be possible to have a CPT representative investor with a maximum tolerable loss of $x_0 = -0.0017$; however, the marginal utility of this investor also could not be used to price all assets since the inequality first-order condition for the third state would not determine its state price.

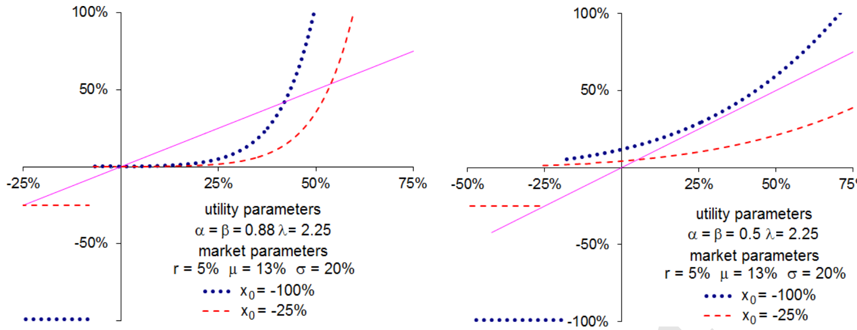

Figure 2: Optimal S-utility portfolios.

Description: The optimal portfolios for investors with S-shaped Tversky-Kahneman utility are plotted against the market portfolio's return. The market has a lognormal distribution with parameters $r = 5\%$, $\mu = 13\%$, $\sigma = 20\%$. The investor's utility parameters are $\alpha = \beta = 0.88$ in the left panel and $\alpha = \beta = 0.5$ in the right panel. In each case $\lambda = 2.25$ and the zero-utility reference return is $\hat{x} = 0$. The maximum tolerable loss is $x_0 = -100\%$ (dotted line) or $x_0 = -25\%$ (dashed line).

return on the market. For an investor with TK's estimated parameters, $\alpha = \beta = 0.88$, the optimal portfolio earns the maximum tolerable loss for market returns up to -6.5% (-8.1%) for x_0 of -100% (-25%). This means the maximum tolerable loss is realized in 18.8% (16.6%) of the years. The Sharpe ratio and Jensen's alpha of the portfolio are also given for purposes of comparison. They are not valid measures of performance for CPT preferences.

The figure shows that the resulting portfolios are quite unusual. For a -100% maximum loss, the portfolio has an average rate of return of 89.1% compared to the market's 13.8% but underperforms the market in 78% of the years. Its average return is higher mostly because in the best years it substantially outperforms the market. It also outperforms the market in years with small losses for the market. The -25% maximum-loss portfolio has a similar record. Its average return is 31.9%, and it requires even better years to outperform the market though, of course, it also beats the market whenever the market drops by more than 25%. What is not shown in the figure is the extreme performance in very good years. The 99$^{\text{th}}$ percentiles returns for these two CPT portfolios are 1681.4% and 563.8%, which occur at 77.1% return for the market. These portfolios are clearly very risky with market betas of 16.4 and 5.4, and standard deviations of 1252% and

Utility Function								
$\alpha = \beta =$	0.88	0.88	0.88	0.88	0.5	0.5	0.5	0.5
max tolerable loss x_0	−100%	−75%	−50%	−25%	−100%	−75%	−50%	−25%
Portfolio Description								
Avg return	89.1%	69.9%	50.9%	31.9%	17.2%	14.1%	10.8%	8.1%
Std. dev. return	1252.4%	973.1%	695.9%	418.5%	34.1%	25.4%	16.2%	8.9%
portfolio's beta	16.40	12.71	9.04	5.37	1.23	0.93	0.61	0.35
% time at x_0	18.8%	18.5%	18.0%	16.5%	5.2%	4.7%	3.5%	1.7%
99th %-tile	1557.3%	1321.8%	945.5%	568.8%	104.4%	82.1%	58.2%	39.5%
Portfolio Performance								
Sharpe ratio	0.067	0.067	0.066	0.064	0.354	0.353	0.348	0.332
Jensen's alpha	−59.1%	−46.0%	−33.0%	−20.0%	1.3%	0.9%	0.4%	−0.1%

Table 1: Optimal CPT portfolios.

Description: This table shows the descriptive and performance statistics of the optimal portfolios for investors with Tversky-Kahneman utility functions with risk parameters $\alpha = \beta = 0.88$ or $\alpha = \beta = 0.5$, $\lambda = 2.25$, $\hat{x} = 0$, and different maximum tolerable losses. The market has a lognormal distribution with a continuously compounded expected rate of return of 13% and a logarithmic standard deviation of 20%. The interest rate is 5%. The market is complete. The state prices are determined assuming a representative investor with a constant relative risk aversion of 2.

418%, respectively; however, these risk measures are strongly affected by the extreme right tail and might not be seen in empirical data.[27]

The right panel shows the optimal portfolio for an investor with $\alpha = \beta = 0.5$, the parameter estimate of Wu and Gonzalez (1996). Though still odd, these portfolios appear more realistic. For example, the betas are now 1.23 and 0.35. Table 1 provides some further summary numbers. One seeming paradox is the positive relation between the magnitude of the maximum tolerable loss, $x_{0,}$ and the fraction of years it is earned. Because a CPT investor is risk seeking over losses, a loss of −100% has less than four times the disutility as a loss of 25%, but the state prices, which are determined by risk aversion, makes the former more than twice as valuable.

It is clear that S-utility can lead to optimal portfolios that are quite different from those held by risk-averse investors. While the portfolios illustrated in Figure 2 and Table 1 assumed complete markets, quite similar portfolios can be constructed with simple put and call options. Optimal CPT portfolios in markets with restrictions on the types of assets are examined later, but first in the next section the second aspect of CPT — probability weighting — is studied.

4 The Portfolio Problem under Cumulative Probability Weighting

In this section, to concentrate on the effects of cumulative probability weighting, some examples using risk-averse utility functions are considered first.[28] For a strictly concave utility function, marginal utility is monotonic and invertible so the optimal portfolio given in equation (12) satisfies

$$x_s^* = u'^{-1}\left((\eta q_s + \kappa_s - \kappa_{s-1})/\omega_s\right). \tag{18}$$

If x_s^* is not equal to either of its neighbors, then $\kappa_s = \kappa_{s-1} = 0$, and the optimal return is the inverse marginal utility of some multiple of the price-decision-weight ratio q_s/ω_s. The only difference between this and the standard result is that the decision weight rather than the probability is used in the ratio. If some consecutive states $s', s'+1, \ldots, s''$ all have the

[27]For example, the 10% Winsorized betas are 5.79 and 1.82.

[28]For these problems there is no loss of generality in assuming the zero-utility reference rate of return is $\hat{x} = 0$.

same return in the optimal portfolio but different returns from those in the other states, then $\kappa_{s'}$ through $\kappa_{s''-1}$ can be positive. In this case,

$$\begin{aligned} x^*_{s'} &= u'^{-1}((\eta p_{s'} + \kappa_{s'})/\omega_{s'}) \le u'^{-1}(\eta p_{s'}/\omega_{s'}) \\ x^*_{s''} &= u'^{-1}((\eta p_{s''} - \kappa_{s''-1})/\omega_{s''}) \ge u'^{-1}(\eta p_{s''}/\omega_{s''}). \end{aligned} \tag{19}$$

The inequalities follow because the multipliers are nonnegative and marginal utility is decreasing. The first-order conditions can therefore be summarized as

$$\begin{array}{ll} x^*_s = u'^{-1}(\eta p_s/\omega_s) & \text{for } x^*_{s-1} < x^*_s < x^*_{s+1} \\ u'^{-1}(\eta p_{s''}/\omega_{s''}) \le x^* \le u'^{-1}(\eta p_{s'}/\omega_{s'}) & \text{for } x^*_{s'} = x^*_{s'+1} = \cdots = x^*_{s''} = x^*. \end{array} \tag{20}$$

The conditions in the second line of (20) permit the optimal portfolio to have tied outcomes across states provided the assumed weak ordering is preserved.

This proposed solution, however, guarantees only an order-constrained optimum. A different ordering of x_s across states might have higher decision-weighted utility. To solve the problem completely, the optimal portfolio for each ordering must be determined from (12), and their maximized decision-weighted utilities, $\sum \omega_s u(x^*_s)$, can then be compared. The optimal portfolio is the constrained portfolio that gives the highest utility.[29]

As noted previously, this solution is optimal only for this particular order-constrained outcome. A complete solution requires checking other possible outcome orderings. For example, consider the simple three-state problem presented in Table 2. The interest rate is zero. States a, b, and c have probabilities of 20%, 30% and 50% and state prices of 0.3, 0.3 and 0.4, respectively. State a is the most expensive state per unit probability, and state c is the least expensive so the optimal portfolio for a risk-averse expected utility maximizer has its returns ordered $x_a < x_b < x_c$.[30] Suppose, instead, the investor has a TK probability weighting function as given in (7) with a parameter $\delta = 0.7$. If his portfolio returns are also ordered $x_a \le x_b \le x_c$, then the decision weights are 25.6%, 20.13% and

[29] As a practical matter the constrained optimal portfolio need not be determined for all orderings. The constraint(s) that are binding in any one of the optimization problems will indicate which orderings to try.

[30] See, for example, Chapter 8 of Ingersoll (1987) for a proof that all risk-averse investors with state-independent utility hold portfolios whose returns are inversely ordered to the price-probability ratio q_s/π_s in a complete market.

54.26%. The optimal order-unconstrained portfolio, determined by (18), is given in the columns labeled "Assumed order (a, b, c)" in the middle panel of Table 2.

Unfortunately, these "optimal" returns are not ordered as assumed, but rather $x_b < x_a < x_c$, and the true decision-weighted utility for this ordering is not 1.065, but only 1.037 as computed with the correct decision weights based on the actual ordering of outcomes as shown in the columns labeled "Corrected (b, a, c)." This occurs because under the first ordering, the decision weights overemphasize state a and underemphasize state b relative to the probabilities so the decision-weight ratio, q_s/ω_s, does not align with probability ratio, q_s/π_s.

But the problem does not end with correcting the computation of the decision-weighted utility. This "optimal" solution was determined using a faulty assumption about the ordering of outcomes, but the result indicated that the $x_a \leq x_b$ constraint was binding. This suggests that the ordering $x_b < x_a < x_c$ be considered. The second panel of Table 2 shows the calculated optimum under this second assumed ordering. Again the order-unconstrained optimal portfolio does not match the assumed order — rather it insists on the originally assumed order of $x_a < x_b < x_c$. Further exploration of all orderings shows that whenever, $x_a < x_b$, increasing x_a and decreasing x_b increases decision-weighted utility and vice versa. Consequently, the optimal portfolio under this probability weighting must have $x_a = x_b$ as shown in the final columns in Table 2. In addition we see that the highest return in state c is larger under probability weighting than under expected utility maximization.

While this example was obviously created, it was not chosen specifically to achieve unusual results; nor do the results depend on the small number of states or on the existence of a complete market. The flattening of the left tail and the skewing of the right tail is a generic trait of the optimal portfolios for investors who employ probability weighting of the type proposed in CPT.

Even when the conversion of probabilities to decision weights is approximately symmetric in the two tails, the effects there are quite different. The decision weights exceed the true probabilities for both extremes as shown in Figure 1. Because the optimal portfolio's return is decreasing in the ratio q_s/ω_s, the portfolio of an investor using probability weighting has higher returns than that of an expected utility maximizer in both tails where ω_s tends to be greater than π_s; that is, the right tail is

Expected-Utility Maximizing Portfolio

state	π	p	p/π	x^*	$\pi \cdot u(x)$
a	20%	0.3	1.5	0.328	0.102
b	30%	0.3	1.0	0.903	0.282
c	50%	0.4	0.8	1.577	0.657
					$\mathbb{E}_\pi[u(\cdot)] = 1.042$

Optimal Decision-Weighted Portfolio

			Assumed order (a, b, c)			Corrected (b, a, c)		Constrained		
state	π	p	$\omega(a, b, c)$	x^*	$\omega \cdot u(x)$	$\omega(b, a, c)$	$\omega \cdot u(x)$	ω	x^*	$\omega \cdot u(x)$
a	20%	0.3	25.60%	0.575	0.184	12.93%	0.093	45.74%	0.437	0.278
b	30%	0.3	20.13%	0.315	0.101	32.81%	0.164			
c	50%	0.4	54.26%	1.832	0.780	54.26%	0.780	54.26%	1.845	0.784
					"$\mathbb{E}_\omega''[u(\cdot)] = 1.065$		$\mathbb{E}_\omega[u(\cdot)] = 1.037$			$\mathbb{E}_\omega[u(\cdot)] = 1.062$

			Assumed order (b, a, c)			Corrected (a, b, c)		Constrained		
state	π	p	$\omega(a, b, c)$	x^*	$\omega \cdot u(x)$	$\omega(b, a, c)$	$\omega \cdot u(x)$	ω	x^*	$\omega \cdot u(x)$
b	30%	0.3	32.81%	0.985	0.325	20.13%	0.210	45.74%	0.437	0.278
a	20%	0.3	12.93%	0.096	0.032	25.60%	0.066			
c	50%	0.4	54.26%	1.689	0.743	54.26%	0.780	54.26%	1.845	0.784
					"$\mathbb{E}_\omega$"$[u(\cdot)] = 1.100$		$\mathbb{E}_\omega[u(\cdot)] = 1.006$			$\mathbb{E}_\omega[u(\cdot)] = 1.062$

Table 2: Optimal portfolio with probability weighting: illustrating portfolio skewing.

Description: This table presents a three-state portfolio for an investor with a utility function $u(x) = x^{0.6}$ who uses a TK probability weighting function with parameter $\delta = 0.7$.

Interpretation: The two sections of the table demonstrate that either assumed ordering of the state's returns (a, b, c) or (b, a, c) leads to a contradiction where the first-order conditions for the optimal portfolio would have the opposite ordering. Therefore, the true optimal portfolio must have equal outcomes in states a and b.

longer and the left tail is shorter, leading to a right skewing of the optimal portfolio.[31]

In the right tail, this stretching is the only effect. In the left tail, however, the increased return can also alter the outcome ordering, which affects the probability weighting as shown in the example in Table 2. As in the example, the left tail will often be completely flattened so that the portfolio's return is constant over a range of the low-return states. This is particularly true for portfolios with many outcomes whose probabilities are similar in magnitude.

Flattening need not occur only in the left tail as the example in Table 3 shows. In this example, the states are ordered a to d from high to low by their ratios q_s/π_s. However, the decision weights flip the ordering of the middle two states b and c for the ratio q_s/ω_s. This alteration would require the optimal portfolio to hold $x_b > x_c$, but this switch in the order also alters the decision weights and the ratio — changing them back to the order under the true probabilities.[32] Therefore, the decision-weighted optimal portfolio will hold $x_a < x_b = x_c < x_d$, and the portfolio outcomes have been flattened in the middle of the distribution not the left tail.

In the two examples in Tables 2 and 3, the portfolio outcomes are monotone decreasing in the price-probability ratio, q_s/π_s; however, they need not be strictly decreasing as they are for any strictly risk-averse expected utility maximizer. It is also possible to construct examples in which the optimal decision-weighted portfolio's returns are not monotonic in the ratio, q_s/π_s, even for a strictly risk-averse investor.

The inverted S-shaped probability weighting function, Ω, increases the importance of both tails of the distribution; therefore, if the ordering of the

[31]It is possible to construct scenarios where the probability weight for either extreme outcome is less than the associated probability, and, therefore, the probability-weighted optimal portfolio has a smaller return than the expected-utility maximizing portfolio for the extreme outcome. For example, this occurs in the lower tail for the TK weighting function with $\delta = 0.65$ if the worst state had a probability in excess of 35.87%. Because the best and worst states are likely to be very rare in most applications, right skewing of both tails of the optimal portfolio should be the typical result. The effect must always be present in at least one of the tails if the cumulative weighting function has an inverted S-shape with only a single crossing of the 45° line.

[32]Any probability weighting function satisfying $\Omega(0.3) = 0.32$, $\Omega(0.5) = 0.51$, $\Omega(0.7) = 0.68$, $\Omega(1) = 1$ will create this example. These conditions are consistent with the inverted-S shape as shown in Figure 1. Note that states b and c have the same probability so the ordering of their outcomes alone determines which decision weight is assigned to which state.

state	π	q	q/π	state order	Ω	ω	q/ω	state order	Ω'	ω'	q/ω'
a	30%	0.40	1.33	a	32%	32%	1.250	a	32%	32%	1.250
b	20%	0.21	1.05	b	51%	19%	1.105	c	51%	19%	1.000
c	20%	0.19	0.95	c	68%	17%	1.118	b	68%	17%	1.235
d	30%	0.20	0.67	d	100%	32%	0.625	d	100%	32%	0.625

Table 3: Optimal portfolio with probability weighting: illustrating midrange flattening.

Description: This table presents a four-state problem for an investor with a risk-aver utility function who uses an inverted S-shaped probability weighting function with $\Omega(0.3) = 0.32$, $\Omega(0.5) = 0.51$, $\Omega(0.7) = 0.68$, $\Omega(1) = 1$.

Interpretation: The middle section of the table demonstrates that a portfolio whose returns are ordered inversely to the likelihood ratio, q/π, will induce a probability weighting likelihood ratio, q/ω, which reverses the order of the middle two states, b and c. However, as shown in the last section, a portfolio whose returns are ordered inversely to the likelihood ratio, q/ω, will induce a new decision-weighted likelihood ratio, $q/\omega\prime$, which again switches the order of the middle two states. Because both assumed orderings lead to contradictions, the true optimal portfolio must have equal outcomes in states b and c.

portfolio's returns moves one state's return from the left to the right tail, the decision weight could remain higher than the probability.

In the example illustrated in Table 4, state c has a lower ratio, q_s/π_s, than state b. Therefore, any risk-averse expected utility maximizer would hold a portfolio earning a higher return in state c than in state b. However, state b is less likely than state c and is further into the left tail than state c is into the right tail. The decision-weighting function, therefore, emphasizes state b relative to state c, and the decision-weight maximizer might wish to increases the return in state b to more than in state c. This alters the ordering and affects the decision weights assigned. In this case, however, state c has a large probability and state b is transferred just as far into the right tail as it was in the left tail so its decision-weight ratio, p_b/ω_b, remains high relative to p_c/ω_c. Therefore, the optimal decision-weighted portfolio has its returns ordered $x_a < x_c < x_b < x_d$, which is not monotonic in the price-probability ratio, q_s/π_s.[33] This will be true for any risk-averse investor.

The examples in Tables 2–4 illustrate some of the complications of finding an optimal decision-weighted portfolio even for concave utility. In any practical problem the difficulty is multiplied immensely as every possible ordering of the state returns might need to be examined. That is, if there are n states, then n factorial standard portfolio problems would need to be solved — one for each of the possible orderings. This exacerbates the portfolio-outcome-ordering problem discussed in connection with the third property of Proposition 3. If investors' portfolios are not aligned, a representative investor holding the market portfolio may not exist. The next section covers this topic in more detail and shows that it can be resolved in some important cases.

5 The Representative Investor under CPT

As noted in the previous two sections, investors who have S-utility or who use decision weights in lieu of actual probabilities may optimally hold portfolios whose outcomes are not aligned even when markets are complete and investors have homogeneous beliefs. This can result in the absence of a representative investor who holds the market portfolio and in

[33] The other permutations must also be checked to verify that this ordering leads to the highest utility.

state	π	q	q/π	state order	Ω	ω	q/ω	state order	Ω'	ω'	q/ω'
a	20%	40%	2.00	a	30%	30%	1.33	a	30%	30%	1.33
b	20%	20%	1.00	b	50%	20%	1.00	c	56%	26%	1.15
c	40%	30%	0.55	c	75%	25%	1.20	b	75%	19%	1.05
d	20%	10%	0.50	d	100%	25%	0.40	d	100%	25%	0.40

Table 4: Optimal portfolio with probability weighting: illustrating non-monotonic response.

Description: This table presents a four-state problem for a risk-averse investor who uses an inverted S-shaped probability weighting function with the properties: $\Omega(0.2) = 0.3$, $\Omega(0.4) = 0.5$, $\Omega(0.6) = 0.56$, $\Omega(0.8) = 0.75$, $\Omega(1) = 1$.

Interpretation: The middle section of the table demonstrates that a portfolio whose returns are ordered inversely to the price-probability ratio, q/π, will induce a probability weighting ratio, q/ω, which reverses the order of the middle two states, b and c. Furthermore, a portfolio whose returns are ordered inversely to the ratio, q/ω, keeps the same decision-weighted ratio ordering. Therefore, the optimal portfolio will have $x_a < x_c < x_b < x_d$ which is not monotonic in the price-probability ratio q/π.

the failure of the standard pricing results, which follow. This section shows that the market portfolio will be optimal in at least one important case — a complete market with equally probable states, including a market with a continuum of states. Two preliminary results are proved first.

Proposition 4 (*Portfolio-Outcome Ordering under Probability Weighting*)**.** *For any two states in a complete market that are equally probable, the optimal portfolio of any investor with increasing utility who uses cumulative probability weighting realizes at least as high a return in the state with the smaller state price.*

Proof. Consider two states, i and j, with $\pi_i = \pi_j$. With no loss of generality take $q_i > q_j$. Now assume that the proposition is false and $x_i^* = h > \ell = x_j^*$. The otherwise identical portfolio with $x_i = \ell$ and $x_j = h$ is affordable because $q_i > q_j$. Swapping these two returns will change the order of the outcomes across states. However, because $\pi_i = \pi_j$ and the weighting function depends on the cumulative probabilities, only the decision weights for states i and j will be affected and they will simply be swapped. Therefore, the decision-weighted expected utility for the altered portfolio will be equal to that for the portfolio originally assumed to be optimal. The altered portfolio costs less by $(q_i - q_j)(h - \ell)$, and this extra can be invested in the risk-free asset (or any other asset with nonnegative payoffs that does not alter the order of the state outcomes) increasing the return realized in every state. With this addition, the final portfolio will have a higher decision-weighted expected utility than the original portfolio, which, therefore, cannot have been optimal. Thus, $x_i^* \leq x_j^*$. $\square$

With the additional assumption that all states are equally likely, Proposition 4 can be strengthened to show that all optimal portfolios are aligned. This replicates the result for any complete market if investors are all risk averse and use objective probabilities.

Proposition 5 **(Weak Monotonicity of Decision-Weighted Portfolio Returns).** *Assume a complete market with equally likely states. Then the returns on the optimal portfolio of any investor with concave or S-utility who uses cumulative probability weighting will be weakly decreasing in the objective price-probability ratio, $\theta = q/\pi$. For risk-averse investors, the returns will be strictly decreasing over ranges where the price-decision-weight ratio, q/ω, is strictly decreasing and be constant over ranges where q/ω is increasing or constant.*

Proof. Weak monotonicity of the returns follows directly from Proposition 4 because all states are equally probable. It only need be determined when the ordering is strict for risk-averse investors.

Order the states by the ratio θ and consider a range of states where q/ω is increasing or constant and assume, contrary to the proposition, that $x_s^* < x_{s+1}^*$. From the first-order conditions in (12), the multiplier κ_s must be zero when the portfolio returns differ so using (18)

$$\begin{aligned} x_s^* &= u'^{-1}\left((\eta q_s - \kappa_{s-1})/\omega_s\right) \geq u'^{-1}\left(\eta q_s/\omega_s\right) \\ x_{s+1}^* &= u'^{-1}\left((\eta q_{s+1} + \kappa_{s+1})/\omega_s\right) \leq u'^{-1}\left(\eta q_{s+1}/\omega_s\right). \end{aligned} \tag{21}$$

The inequalities follow because the remaining two multipliers are nonnegative and u'^{-1} is a decreasing function. But the monotonicity of u'^{-1} also implies that

$$x_{s+1}^* \leq u'^{-1}\left(\eta q_{s+1}/\omega_s\right) \leq u'^{-1}\left(\eta q_s/\omega_s\right) \leq x_s^* \tag{22}$$

which is a contradiction so $x_s^* = x_{s+1}^*$ when q/ω is increasing or constant.

Now consider a range where q/ω is decreasing and assume, contrary to the proposition, that $x_s^* = x_{s+1}^*$. Suppose the portfolio is altered by earning ε less in state s and $q_s\varepsilon/q_{s+1}$ more in state $s+1$. This altered portfolio has the same cost as the original, and changes the expected decision-weighted utility by

$$\begin{aligned} \Delta\mathbb{E}_\omega[v(x)] &= \omega_s[v(x-\varepsilon) - v(x)] + \omega_{s+1}[v(x + q_s\varepsilon/q_{s+1}) - v(x)] \\ &\approx v'(x)q_s\varepsilon\left[\omega_{s+1}/q_{s+1} - \omega_s/q_s\right] > 0 \end{aligned} \tag{23}$$

which is positive because ω/q is increasing. Again this is a contradiction so $x_s^* < x_{s+1}^*$ when q/ω is decreasing. □

We have already seen in Table 4 that this monotonicity result need not obtain in a complete market with states having different probabilities. However, it can be extended to such markets provided investors can create financial contracts that are fair-value sub-state bets. When such financial contracts can be created, any state with probability, π, and state price, q, can be partitioned into two or more sub-states with proportional probabilities and sub-state prices; i.e., $q'/\pi' = q/\pi$ for all sub-states of the same original state. Proposition 5 can be applied to these equally probable sub-states and then extended back to the original states by aggregation.[34]

[34] In some cases no equally probable subdivision of states is possible with a finite number of sub-states. For example, when any state has an irrational probability, the equally

Figure 3 illustrates the three ratios. The ratio q/π is falling (by construction). The ratio ω/π is U-shaped because the decision weights are larger than the actual probabilities for the outcomes in both tails. The ratio q/ω is the quotient of the two and has an inverted U-shape. It must be decreasing in the range where ω/π is rising, but is increasing when ω/π is sufficiently steeply declining. The typical case is illustrated with q/ω increasing for the smallest values of the market's return. In such an economy the optimal portfolio of a risk-averse decision-weight maximizer will have the same return in all of the poorest outcome states and will have returns increasing with the state in the better states, like that of an objective-expected utility maximizer. The right-hand panel of Figure 3 shows the optimal complete-market portfolio of an investor who uses a Tversky-Kahneman weighting function. The investor has a constant relative risk aversion of 2 and, if using objective probabilities, would hold the market. For the TK weighting function with $\delta = 0.9$, the investor's portfolio is constant with a loss of 19.7% whenever the market loses more than 42.2%. For $\delta = 0.6$, the return is a constant loss of 15.5% whenever the market loses or has a gain less than 0.2%. This aspect of the optimal portfolio resembles portfolio insurance, though the leverage on the upside increases rapidly the higher is the market return. At a market return of 25%, the two portfolios' slopes are 1.3 and 2.7; at a market return of 50%, the slopes are 1.8 and 6.2.

An important implication of Proposition 5 is that with homogeneous objective beliefs, the market portfolio itself will be objectively efficient in a complete market. This means that a representative investor exists, and all the strong intuitions that follow from this representation will remain available.

Proposition 6 (Objective Efficiency of Market Portfolio under Probability Weighting). *Assume homogeneous objective beliefs, a complete market with equally likely states, all investors are risk averse or have S-utility. In addition, there is at least one risk-averse investor who uses objective probabilities. Then, in equilibrium, the market portfolio's returns will be*

probable state partitioning must divide that state into m sub-states each with probability π/m and must create in total n sub-states each with probability $1/n$ across all the original states. But $\pi/m \neq 1/n$ when π is irrational. Of course, it will always be possible to construct states that are equally likely to any desired accuracy as $n \to \infty$. And the proposition as well as Proposition 6 below can be directly applied in a model with a continuum of states and a continuous probability density.

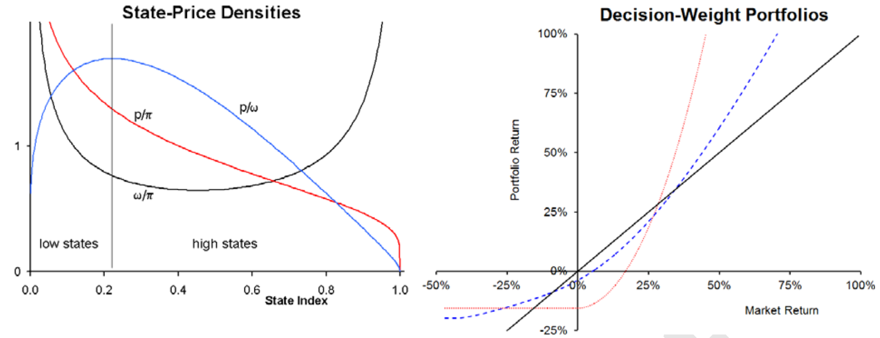

Figure 3: State price probability ratios and optimal state price to decision weight ratio.

Description: This figure illustrates the ratios of the state price to the true state probabilities, q/π, and the state price to the decision weights, q/ω.

Interpretation: The former ratio is decreasing by assumption. The ratio, q/ω, is typically increasing first, and decreasing later. Optimal decision weight portfolio returns are constant for the increasing portion and increasing when q/ω is decreasing. In the right-hand panel, the market has a lognormal distribution with parameters $r = 5\%$, $\mu = 13\%$, $\sigma = 20\%$. The cumulative decision weighting function is that proposed by Tversky and Kahneman with $\delta = 0.9$ (dotted line) and $\delta = 0.6$ (dashed line).

strictly decreasing in the price-probability ratio, and the market portfolio will be objectively risk-averse efficient.

Proof. From Proposition 5, the returns on each investor's optimal portfolio are weakly ordered inversely to the objective price-probability ratio. Because the market portfolio is a convex combination of these optimal portfolios, its returns must also be weakly decreasing in the ratio. Now assume this monotonicity is not strict; that is, assume there are two states with different price-probability ratios but equal market returns. The risk-averse investor using objective probabilities does hold a strictly monotone portfolio with a higher return in the better state; therefore, if the market is to clear with an equal return in the two states, some other investor must hold a portfolio with a smaller return in the better state. But this contradicts Proposition 5. So in equilibrium, the market portfolio's returns must be strictly decreasing in the objective price-probability ratio and therefore optimal for some strictly risk-averse utility function. □

Proposition 6 shows that the market is an objectively risk-averse efficient portfolio; that is, the market portfolio is the optimal portfolio for some strictly risk-averse investor who does not use probability weighting to modify the correct objective beliefs. This is significant because it assures the existence of a risk-averse representative agent who uses the objective probabilities. This means that price data alone cannot logically reject a risk-averse objective-probability (*classical*) equilibrium in favor of a (*behavioral*) equilibrium arising from S-utility or probability weighting. However, this statement has several caveats. First, the proposition does not prove that there is a unique representative agent so there may also exist one (or more) S-utility representative agents who do use probability weighting, and this latter representation might be viewed as statistically more likely. Second, other information such as asset holdings, trades, volume, etc. might be inconsistent with a classical equilibrium. Third, this is a single-period result so there might be evidence in price dynamics that are inconsistent with a classical equilibrium. Finally, all of the analysis assumes homogeneous objective beliefs.

In this paper, only the first caveat will be examined. How do prices differ between a standard and behavioral equilibrium. CAPM-like predictions cannot be made without many more assumptions about the joint return distribution of the individual assets and the market, but the effects of CPT on market derivatives can be examined. To illustrate, suppose the market portfolio has an objective lognormal distribution with a logarithmic volatility of 20% and a risk premium of 8% in excess of the 5% interest rate. As shown in Rubinstein (1976), the market will be the optimal portfolio for an investor with a constant relative risk aversion of 2, and call option prices will be given by the Black-Scholes model. If the representative investor has S-utility or uses probability weighting, option prices will deviate from those values. This is shown in Figures 4 and 5, which plot the implied volatilities of various options.

In Figure 4, the representative investor has constant relative risk aversion but uses TK probability weighting. The effect is straightforward. A small probability weighting parameter, δ, puts more emphasis on the tails increasing the subjective variance relative to the objective variance. This increases the option's implied volatility. The effect is approximately the same for all options being only slightly larger for near-the-money options whose prices are most sensitive to variance.

In Figure 5, the representative investor uses objective probabilities, but has a TK utility function. This S-utility produces an obvious volatility smile as is seen in actual option prices; however, the high-strike portion of the

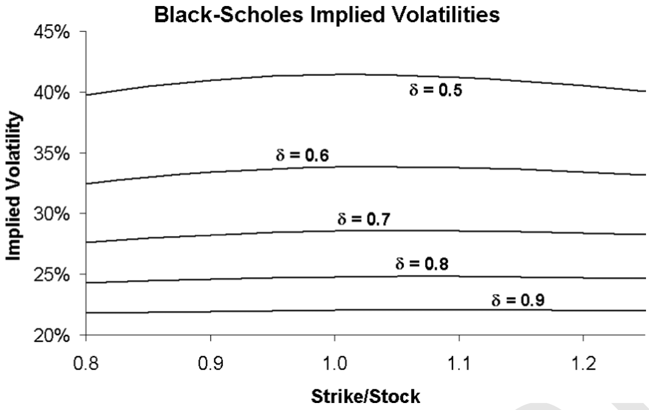

Figure 4: Implied option volatilities under decision weighting.

Description: This figure shows the Black-Scholes implied volatility for call options written on the market portfolio with various strike prices. The market portfolio has a lognormal distribution with a logarithmic variance of 20%. The option prices are determined assuming a representative investor with constant relative risk aversion and a TK probability weighting function with parameter δ.

smile is more pronounced which is atypical. Implied volatilities are also significantly lower than the true volatility; though, as seen in Figure 4, probability weighting would offset this result. The smile is deeper and implied volatilities are lower the larger is the utility function's curvature, α and β. The loss aversion parameter, λ, has only a minor effect on the level of the implied volatility.

Proposition 6 remains valid even if the market is apparently incomplete, provided investors are unconstrained in the types of financial contracts they can introduce. If the introduction of financial assets is unrestricted but the market remains apparently incomplete, the shadow prices of any financial assets that have not been created must be the same for all investors otherwise it would benefit those investors to introduce such contracts. In particular, they must agree on the state prices for all states even if all pure state securities cannot be constructed from the exiting assets. If pure state financial securities were introduced at these shadow prices, the gross demand for them would be zero, and the equilibrium would remain unchanged.[35]

[35]Because S-utility investors are risk seeking over losses, they would typically "more than complete" the market introducing financial assets that provided pure within-state bets

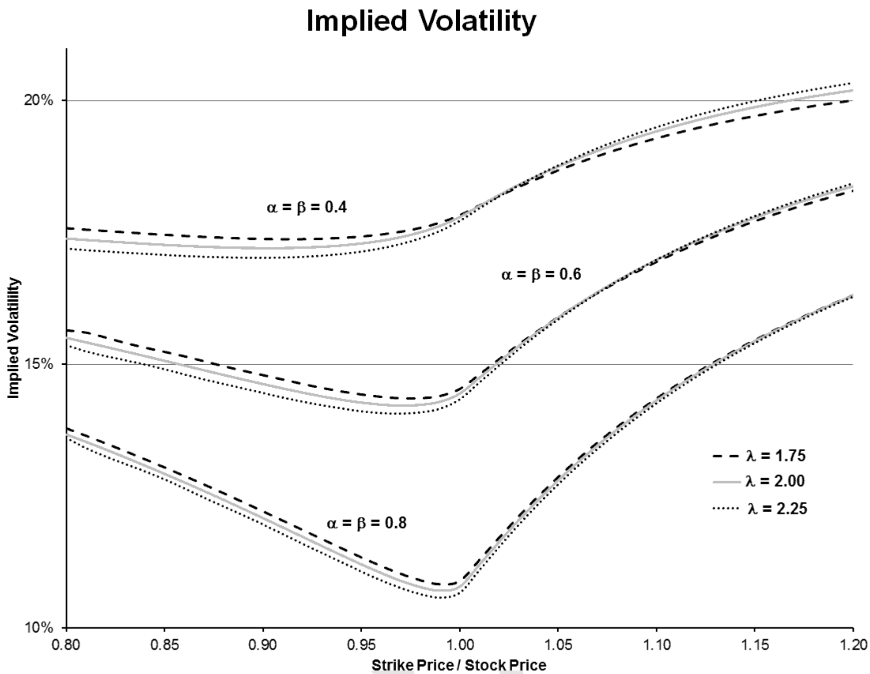

Figure 5: Implied option volatilities with S-utility.

Description: This figure shows the Black-Scholes implied volatility for call options on the market portfolio with various strike prices. The market portfolio has a lognormal distribution with a logarithmic variance of 20%. The option prices are determined assuming a representative investor with TK utility with parameters α, β, and λ.

If market completion with financial assets is not allowed, then the inverse ordering between optimal portfolios and the ratio, q/π, need not hold. Of course, that property need not hold amongst investors using the true probabilities either. The portfolio problem in an incomplete market can be analyzed as above by adding constraints, but little can be said in general. The next two sections of the paper examine two incomplete markets, which have been analyzed extensively, markets displaying two-fund separation and those in which mean-variance analysis is applicable.

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as well. These claims serve to concavify their utility functions; the representative investor would have linear utility for losses and concave utility for gains.

6 Cumulative Prospect Theory and Mutual Fund Separation

Outside of complete markets, the most commonly analyzed market structure is one in which mutual fund separation holds — in particular, the mean-variance model of two-fund separation. Two-fund separation is of considerable interest in finance because it yields strong predictions with sound intuition in a tractable setting. Under two-fund separation,[36] the set of optimal portfolios is spanned by the risk-free asset and a single risky portfolio that is, perforce, the market portfolio of risky assets. A complete market ensures the existence of a representative investor who holds the market. Two-fund separation goes beyond that. It is essentially a construction of the representative investor identifying the exact first-order condition that prices all assets in relation to the market portfolio. There are two types of two-fund separation. The first holds when all investors' utility functions are from the linear-risk-tolerance (LRT) class with the same cautiousness (see Cass and Stiglitz, 1970), the second when all asset distributions come from the separating distributions (see Ross, 1978).

Utility-based two-fund separation will not obtain under CPT. Clearly the S-utility functions with both concave and convex portions are not of the necessary LRT class. Nor, will utility-based two-fund separation hold even for LRT investors who use probability weighting unless they have identical probability weights. While this could be coincidentally true, it would typically only arise if investors had homogeneous objective beliefs and used the same weighting functions. For example, mean-variance analysis remains valid for quadratic utility under probability weighting. All portfolios can be ranked by just knowing their mean and variance, but the means and variances required are those computed using decision weights. Therefore, investors will have different (decision-weighted) mean-variance efficient frontiers if they use different probability weighting functions even if they have homogeneous objective beliefs. Their optimal portfolios will no longer be the same except for leverage, and the CAPM equilibrium will not result.

Distributional-based separation also cannot hold in general with S-utility or probability weighting. Ross (1978) has shown that investors hold

[36] In the absence of a risk-free asset, two-risky-fund separation can still hold for certain restrictions on utility or probability distributions or both; see Cass and Stiglitz (1970) and Ross (1978). Throughout this paper only two-fund money separation, when one of the two mutual funds is the risk-free asset, is considered.

combinations of a single risky-asset index portfolio and the risk-free asset if and only if returns are characterized by

$$\begin{aligned} &\tilde{r}_i = r_f + b_i \tilde{y} + \tilde{\varepsilon}_i \\ &\text{with } \mathbb{E}[\tilde{\varepsilon}_i | y] = 0 \quad \forall i \\ \exists \boldsymbol{\alpha} \quad &\text{such that } \mathbf{1}'\boldsymbol{\alpha} = 1, \boldsymbol{\alpha}'\tilde{\varepsilon} \equiv 0. \end{aligned} \tag{24}$$

Under the conditions in (24), all risk-averse investors optimally hold some mixture of the risk-free asset and the index portfolio, $\boldsymbol{\alpha}$, with excess return $\tilde{y}$, which has no residual risk. The optimality of these mixtures follows immediately by second-order stochastic dominance. These conditions are clearly necessary for two-fund separation under CPT, which includes an identity weighting function and piecewise linear, concave utility, but they are not sufficient.

An investor with S-utility might not choose the stochastically dominating index portfolio if some other portfolio has nonzero residual risk only when $\tilde{y}$ is sufficiently below its mean. In this case, the gains on the two portfolios can have the same distribution, but the extra riskiness of the losses on the portfolio with residual risk can lead to its preference by S-utility investors because they are risk seeking with respect to losses. So an S-shaped utility function destroys Ross' two-fund separation result even with no probability weighting.

Probability weighting also extinguishes two-fund separation even among risk-averse investors. For example, suppose $y = \{-1, 2\}$ with equal probability, $r_f = 1$, and an asset with $b = 1$ has $\varepsilon \equiv 0$ when $y = -1$ and $\varepsilon = \pm 1$ with equal probability when $y = 2$. The residual risk ε is conditionally mean zero, as required, making the asset riskier than the index. Consider a risk-averse investor with the concave utility function $u(w) = w$ for $w > 0$ and $u(w) = 3w$ for $w \leq 0$ and a probability weighting function that assigns $\Omega(0.5) = 0.5$, $\Omega(0.75) = 0.70$. This investor will compute an expected payoff and utility of 1.6 for the asset and 1.5 for the index. De-levering the index with lending decreases its expected return and utility. Levering the index with borrowing increases its expected return but also decreases its expected utility, as the bad outcome moves into the high marginal utility region. So this investor's optimal portfolio cannot be a levered position in the index, and two-fund separation does not hold.

This example illustrates the complexity of establishing two-fund separation under probability weighting. Two-fund separation requires showing that for any increasing, concave utility function and any portfolio in a large class, there exists a portfolio in a smaller class (those that have no idiosyncratic risk) that gives at least as high an expected utility. Using objective probabilities, any portfolio in Ross' class with a given b has the same expected return as and is more risky than the levered position in the index with the same b. Because this specific levered index portfolio stochastically dominates all assets with the same b value, each utility function need not be considered separately — two-fund separation holds trivially. However, with probability weighting, a levered position in the index no longer dominates all other portfolios with the same b because the convex portion of an inverted-S-shaped weighting function can increase the subjective mean after an objective-mean-preserving spread. Therefore, risk-neutral investors and those sufficiently close to risk neutral will prefer the portfolio that is objectively dominated. This does not mean that two-fund separation fails, but it does mean that to verify separation just by comparing portfolios with the same leverage b is insufficient. Every portfolio with idiosyncratic risk must potentially be compared to all levered index positions with the same or higher subjective mean.

Although Ross' two-fund separation does not hold, two related questions immediately arise. Are there weighting functions and restrictions on utility that do preserve two-fund separation for Ross' distributions? Can the class of separating distributions be further restricted so that mutual fund separation does hold for some or all S-utility functions and inverse-S weighting functions? In fact, the first question has already mostly been answered by the previous examples. Two-fund separation cannot hold for all of the distributions in Ross' class whenever the set of utility functions considered includes any with any strictly convex portion because a portfolio with residual risk in only that region will be preferred to the same portfolio with no residual risk. Similarly any strictly convex portion of the probability weighting function increases the subjective mean of some objective-mean-preserving spreads eliminating the second-order stochastic dominance.

Ross' two-fund separation, therefore, can remain valid only for risk aversion and concave weighting functions. Proposition 7 shows that these conditions are sufficient as well.

Proposition 7 (Two-Fund Separation with Concave Probability Weighting). *Ross' two-fund separation result holds for risk-averse investors if and only if all investors have weakly concave probability weighting functions.*

Proof. The necessity of concavity of the weighting function has already been discussed. The Ross conditions in equation (24) are sufficient for two-fund separation under probability weighting if the weighting function preserves the second-order stochastic dominance inherent in the objective probabilities.

Levy and Kroll (1979) show that two of the equivalent definitions of second-order stochastic dominance of the distribution F over the distribution G are

$$\begin{aligned} &\mathbb{E}_F[u(x)] \geq \mathbb{E}_G[u(y)] \quad \forall u \text{ with } u' \geq 0,\ \ u'' \leq 0 \\ &\Leftrightarrow \ \ 0 \leq \int_0^P [F^{-1}(p) - G^{-1}(p)]dp \quad \forall P. \end{aligned} \tag{25}$$

Second-order stochastic dominance is preserved under the weighting function, Ω, if

$$\mathbb{E}_{\Omega(F)}[u(x)] \geq \mathbb{E}_{\Omega(G)}[u(y)] \Leftrightarrow 0 \leq \int_0^P [F^{-1}(\Omega^{-1}(p)) - G^{-1}(\Omega^{-1}(p))]dp \forall P. \tag{26}$$

Using the change in variable, $p \equiv \Omega(q)$, the integral in (26) can be re-expressed as

$$\begin{aligned} &\int_0^{\Omega^{-1}(P)} [F^{-1}(q) - G^{-1}(q)]\Omega'(q)dq \\ &\quad = H(q)\Omega'(q)\Big|_0^{\Omega^{-1}(P)} - \int_0^{\Omega^{-1}(P)} H(q)\Omega''(q)dq \end{aligned} \tag{27}$$

where

$$H(Q) \equiv \int_0^Q [F^{-1}(q) - G^{-1}(q)]dq.$$

The first term on the right-hand side of (27) is nonnegative because $H(0) = 0$ and H and Ω' are nonnegative elsewhere. The remaining integral is nonpositive as H is nonnegative and Ω'' is non-positive. Therefore the integral in (26) is nonnegative and stochastic dominance is preserved. ☐

Recall that the decision weight density is the product of the probability density and the derivative of the weighting function. A concave weighting function has a decreasing derivative so it emphasizes the probabilities of the low payoffs relative to those of high payoffs; therefore, $\mathbb{E}_\Omega[\tilde{\varepsilon}] \leq$

$\mathbb{E}[\tilde{\varepsilon}] = 0$. So any mean-preserving spread, ε, adds risk and cannot increase the expectation. This ensures that objectively stochastically dominated prospects remain stochastically dominated under probability weighting.

To preserve two-fund separation under inverse-S weighting functions as utilized in CPT rather than just concave weighting functions, objective-mean preserving spreads that increase the subjective mean must be precluded. One way to accomplish this is to assume enough symmetry so that any mean-increasing alteration in one tail has an offsetting mean-reducing alteration in the other tail. This requires symmetric distributions and "no better than symmetric" probability weighting adjustments.

Definition 2. *A probability weighting function, Ω, for a cumulative distribution, F, is a symmetric Quiggin weighting (SQW) if it is strictly increasing, concave below 1/2, and complementary around 1/2 with $\Omega(1-F) = 1 - \Omega(F)$.*[37] *It is a (strictly) concavified symmetric Quiggin weighting (CSQW) if $\Omega(F) \equiv \Psi(\Xi(F))$ where $\Xi(\cdot)$ is an SQW, and $\Psi(\cdot)$ is strictly increasing and (strictly) concave.*

As its name implies, an SQW ensures that a symmetric distribution will remain symmetric after it is applied by adjusting the two tails in the same way. It is the same as applying the same weighting function separately to gains and losses with the restriction that an objective probability of 1/2 is always mapped to a subjective probability weight of 1/2. When a strictly CSQW is applied, it introduces a form of pessimism weakly increasing every percentile. When applied to a symmetric distribution, the probability weight that a loss exceeds some size becomes larger than the probability weight that a gain exceeds the same size.

Proposition 8 (Two-Fund Separation under CSQW). *Sufficient conditions for two-fund separation under risk aversion and probability weighting are: (i) returns satisfy the Ross conditions for two-fund separation as given in (24); (ii) the distributions of $\tilde{y}$ and all asset returns, $\tilde{r}_i$, are symmetric;*[38] *and (iii) the probability weighting function is a CSQW.*

[37] Quiggin (1982) originally proposed that weighting functions should be increasing, concave (convex) below (above) 1/2, and that $\Omega(1/2) = 1/2$. The only additional property required here is complementarity symmetry. The symmetry restriction ensures that $\Omega(1/2) = 1/2$ and that Ω is convex above 1/2. Note that the identity function (i.e., using objective probabilities) is a SQW and any concave weighting function is a CSQW.

[38] Assumption (ii) does not require that the distributions of $\tilde{\varepsilon}$ be symmetric. If the distributions of $\tilde{\varepsilon}$ conditional on $\tilde{y} = \bar{y} + a$ and that of $-\tilde{\varepsilon}$ conditional on $\tilde{y} = \bar{y} - a$ are identical, then the asset returns will be symmetric.

Proof. Let F and G be the cumulative distributions of $r_f + \tilde{y}$ and $r_f + b_i\tilde{y} + \tilde{\varepsilon}_i$. Because F and G are the distributions of symmetric random variables, the symmetric transformations $\Xi(F)$ and $\Xi(G)$ preserve the riskiness ordering as shown in Lemma 3 in the Appendix. Therefore, $\Xi(G)$ is also subjectively riskier than $\Xi(F)$ in a Rothschild-Stiglitz sense. Now applying Proposition 7, the increasing, concave transformation Ψ preserves the second-order stochastic dominance. □

Unfortunately assumption (iii) does not apply to the TK weighting function for any parameter value nor to many of the other probability weighting functions used in CPT. The second derivative of a CSQW is $\Omega'' = (\Xi')^2\Psi'' + \Xi''\Psi'$, and because $\Psi'' \leq 0 < \Psi'$, its inflection point can only occur where Ξ' is positive which, by assumption, is for an objective probability in excess of one-half. However, the inflection point for the TK probability weighting function occurs at a probability less than one-half for all values of δ and occurs at $1/e$ for all parameter values of α for Prelec's preferred single parameter function ($\beta = 1$).[39] In fact, using a non-parametric approach to determine the "least favored" probability, Wu and Gonzalez (1996) have estimated that the test subjects' inflection points are no higher than 40%. This does not mean that the typical weighting functions will necessarily induce investors to seek out symmetric objective risks, but only that it cannot be precluded they will not do so. Their actions depend on their utility functions as well.

Furthermore, even the assumed symmetry of the distributions in Proposition 8 is insufficient to guarantee two-fund separation with S-utility. The symmetry assures that risk in both the upper and lower tails is the same, but to preserve the separation, the bad upper-tail risk has to more than offset the good lower-tail risk. Consider an asset or portfolio that has a small amount of residual risk ($\sigma_\varepsilon^2 \approx 0$), which is nonzero only at two isolated points of the index y. By symmetry these two points must be equally distant from the mean,$\bar{y} \pm a$.[40] Because the levered index and the asset have the same return except at those two points of the index

[39]The inflection point for Prelec's two-parameter function can be at probabilities above one-half for some parameter values; e.g., $\alpha = 0.9$, $\beta = 0.7$.

[40]With no loss of generality, it can be assumed that $\bar{y} > 0$ because $-y$ can equally well serve as the random variable describing the index. In any case investors will hold an index with a symmetric distribution in preference to the risk-free asset only if it has a positive risk premium.

return and the probability of those two points are equal by symmetry, the difference between the index's expected utility and that of the asset is

$$\begin{aligned} &v(r_f + b\bar{y} + ba) + v(r_f + b\bar{y} - ba) \\ &\qquad - \mathbb{E}_\varepsilon[v(r_f + b\bar{y} + ba + \tilde{\varepsilon}) + v(r_f + b\bar{y} - ba + \tilde{\varepsilon})] \\ &\qquad \approx -\tfrac{1}{2}[v''(r_f + b\bar{y} + ba) + v''(r_f + b\bar{y} - ba)]\sigma_\varepsilon^2, \end{aligned} \tag{28}$$

with the approximation from a second-order Taylor expansion. To guarantee two-fund separation, the bracketed term must be negative for all possible comparisons. If both arguments of v'' are positive this is true, but they can have mixed signs as well with the negative argument being smaller in magnitude. Therefore, two-fund separation is guaranteed to hold only if $v''(-x_1) + v''(x_2) \leq 0$ for all $0 < x_1 < x_2$. Unfortunately, this condition cannot hold for any utility function defined for all positive outcomes that is increasing, twice-differentiable, and with a strictly convex loss portion[41] so Ross' two-fund separation cannot apply to such a class even with symmetric distributions.

The problem of extending two-fund separation to CPT still remains. Solving this problem with S-utility requires an assumption stronger than symmetry. Solving it with probability weighting requires comparing portfolios with different levels of both systematic and idiosyncratic risk and not just the latter. Stronger distributional assumptions that allow the comparison of all portfolios are required. One answer to this problem is mean-variance analysis — the same hypothesis that simplifies portfolio comparison under objective probabilities. As shown in the next section, two-fund separation and a resulting CAPM equilibrium do hold in many cases under CPT conditions.

7 Mean-Variance Analysis under CPT

Mean-variance analysis is a complete description of a portfolio problem when utility is quadratic or when asset returns are drawn from the class of elliptical distributions.[42] Obviously a quadratic function cannot have the

[41]If $v''(-x_1) = c > 0$, then the condition requires that $v''(x) \leq -c \forall x > x_1$. But if the second derivative is bounded away from zero, the first derivative cannot remain positive as x increases without bound. Mutual fund separation therefore requires some additional conditions based both on the utility functions and an upper bound on asset returns.

[42]See Chamberlain (1983), Owen and Rabinovitch (1983), and Ingersoll (1987) for more details about elliptical distributions and their application to the mean-variance portfolio problem.

S-shape desired under CPT. A two-piece quadratic function can have this shape, but expected utility is not completely described by the mean and variance for such utility functions.[43] However, most of the mean-variance results of elliptical distributions, including the CAPM, continue to be valid with S-shaped utility and probability weighting.

Elliptical distributions get their name from the shape of their iso-probability manifolds. The best-known example of an elliptical distribution is the multivariate normal. More generally, an elliptical distribution for a vector of n rates of return $\mathbf{r}$ is prescribed by its characteristic function, Φ, and its probability density, g, if the latter exists.

$$\begin{aligned}\Phi(\mathbf{t};\boldsymbol{\mu},\boldsymbol{\Theta}) &\equiv \mathbb{E}[e^{i\mathbf{t}'\mathbf{r}}] = e^{i\mathbf{t}'\boldsymbol{\mu}}\varphi(\mathbf{t}'\Theta\mathbf{t}) \\ g(\mathbf{r};\boldsymbol{\mu},\boldsymbol{\Theta},n) &= k_n|\boldsymbol{\Theta}|^{-1/2}h\left((\mathbf{r}-\boldsymbol{\mu})'\boldsymbol{\Theta}^{-1}(\mathbf{r}-\boldsymbol{\mu})\right),\end{aligned} \tag{29}$$

where $\boldsymbol{\mu}$ is the vector of means and $\boldsymbol{\Theta}$ is the covariance matrix.[44] The function h can be any scalar function from the nonnegative reals to the nonnegative reals that can be normalized to give unit mass; k_n is a normalizing constant. For example, in the multivariate normal $h(q) = e^{-q/2}$, and $k_n = (2\pi)^{-n/2}$.

Elliptical distributions have two properties important for portfolio analysis. Every portfolio is completely characterized by its mean and variance and higher mean is preferred. These properties along with conditions under which variance is disliked are given in the next proposition.

Proposition 9 (Mean-Variance Characterization for CPT). *For any elliptical distribution, every portfolio is completely characterized by its objective mean and variance. For a fixed objective variance, all investors with increasing utility prefer a higher objective mean. If an investor has strong loss aversion and modifies probabilities with a CSQW, then variance is disliked on any*

[43]Using the pieced quadratic utility $u(x) = x - bx^2$ for $x \geq 0$ and $u(x) = -\lambda(x - cx^2)$ for $x < 0$, four moment parameters are required to express expected utility. The simplest set is $\mathbb{E}[x|x \geq 0]$, $\mathbb{E}[x^2|x \geq 0]$, $\mathbb{E}[x|x < 0]$, and $\mathbb{E}[x^2|x < 0]$. If probability weighting is also applied, then these must be the probability-weighted expectations, which generally will differ among investors even if they have homogeneous objective beliefs.

[44]If there is a risk-free asset, it can be included separately in the usual fashion. $\boldsymbol{\Theta}$ is non-singular (after removing any redundant risk-free asset) to prevent arbitrage. Elliptical distribution can be fat-tailed with undefined variances or even means, e.g., the multivariate Cauchy distribution. Because all elliptical distributions are symmetric, $\boldsymbol{\mu}$ is always the vector of medians, and $\boldsymbol{\Theta}$ is a general co-dispersion matrix even if means or variances are undefined. The discussion below remains valid and the CAPM holds in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\Theta}$ for such elliptical distributions, provided, of course, that expected utility is defined.

portfolio with a positive subjective mean ($\mu_P > \hat{x}$), and the investor's objective assessment function $J_\Omega(\mu,\sigma) \equiv \mathbb{E}_\Omega[v(\tilde{r}_P(\mu,\sigma) - \hat{x})]$ is quasiconcave.

Proof. It is well known that all linear combinations of elliptical variables are completely characterized by their mean and variance. This can be verified immediately from (29). Define the vector $\mathbf{t} \equiv t\boldsymbol{\alpha}$, then the characteristic function of the return on any portfolio $\boldsymbol{\alpha}$ with mean, $\mu_P = \boldsymbol{\alpha}'\mu$, and variance, $\sigma_P^2 = \boldsymbol{\alpha}'\boldsymbol{\Theta}\boldsymbol{\alpha}$, is $\mathbb{E}[\exp(it\boldsymbol{\alpha}'\mathbf{r})] = \exp(it\mu_P)\varphi(t^2\sigma_P^2)$, Therefore, a portfolio has the univariate probability density, $g(r_P;\mu_P,\sigma_P^2,1)$, and its rate of return can be expressed as a translated and scaled variable $\tilde{r}_P = \mu_P + \sigma_P\tilde{\rho}$ where $\tilde{\rho}$ is a standardized elliptical variable with zero mean and unit variance.

Define the derived objective mean-variance assessment function for a particular weighting function, Ω, as

$$J_\Omega(\mu,\sigma) \equiv \mathbb{E}_\Omega[v(\tilde{r}_p(\mu,\sigma) - \hat{x})] \equiv \int_{-\infty}^{\infty} v(X(\rho))d\Omega(F(\rho)) \qquad (30)$$

where $X(\rho) \equiv \mu + \sigma\rho - \hat{x}$, and F is the univariate cumulative distribution for ρ. $X(\rho)$ is strictly increasing in μ, v is strictly increasing, and $d\Omega(F(\rho))$ is nonnegative; therefore, J_Ω must be strictly increasing in μ.

For the assumed form of the weighting function, define the subjective density for ρ as

$$\begin{aligned} d\Omega(F(\rho)) &= d\Psi(\Xi(F(\rho))) \\ &= \Psi'(\Xi(F(\rho)))\Xi'(F(\rho))dF(\rho) \equiv \psi(\rho)\xi(\rho)f(\rho)d\rho. \end{aligned} \qquad (31)$$

Ψ is strictly increasing and concave so its derivative, ψ, is positive and decreasing. Ξ is an SQW so its derivative, ξ, is an odd function with $\xi(-\rho) = \xi(\rho)$. Substituting (31) into (30), differentiating, splitting the integral at $\rho = 0$, and combining the two parts gives

$$\frac{\partial J_\Omega(\mu,\sigma)}{\partial\sigma} = \int_0^{\infty} \rho\xi(\rho)f(\rho)[v'(X(\rho))\psi(\rho) - v'(X(-\rho))\psi(-\rho)]d\rho < 0. \qquad (32)$$

The first three factors in the integrand are all positive, so the sign of $\partial J_\Omega/\partial\sigma$ is the same as the sign of the term in square brackets. As previously noted, ψ is positive and decreasing so $\psi(-\rho) \geq \psi(\rho) > 0$. In addition, $v'(X(-\rho)) > v'(X(\rho)) > 0$ when the portfolio's mean exceeds

$\hat{x}$ so the bracketed term and $\partial J/\partial\sigma$ must both be negative, meaning variance is strictly disliked on all portfolios with a positive subjective mean.[45]

The signs of the two partial derivatives ensure that over the range of positive subjective means, the μ-σ indifference curves of J_Ω are strictly increasing and the function is quasiconcave. □

Because the derived utility function, J_Ω, is quasiconcave, and the set of feasible portfolios is convex in μ-σ space, standard optimization techniques apply. Proposition 9 together with homogeneous objective beliefs and the usual no-market-frictions assumptions is more than required to prove that any resulting equilibrium is the CAPM. However, all this is not quite sufficient to prove that an equilibrium exists. Stated formally:

Proposition 10 (*Objective CAPM under CPT*)**.** *Assume (i) the returns on all assets are elliptically distributed, (ii) all investors have strictly increasing utility, homogeneous objective beliefs, a common single-period horizon, and evaluate outcomes using a strictly increasing and once differentiable probability weighting function, (iii) there are no transactions costs or differential taxes, (iv) borrowing and lending (or if there is no risk-free asset, short sales) are unrestricted, and shares are infinitely divisible. Then, provided an equilibrium exists, two-fund separation and the CAPM relation between the objective means and covariances will obtain.*

Proof. The proof follows immediately from Equation (30) of Proposition 9. Because J_Ω is strictly increasing in μ, all optimal portfolios must be on the upper limb of the objective minimum-variance hyperbola (if there is no risk-free asset) or on its tangent borrowing-lending line (if there is). In either case, the set of optimal portfolios is spanned by any two portfolios it includes. If an equilibrium exists, then the market portfolio is a convex combination of optimal portfolios and is mean-variance efficient. The relevant objective CAPM equilibrium results in the usual fashion. □

Of course, the CAPM of this proposition need not be the same as would prevail if all investors used objective probabilities and were risk averse.

[45]For those values of ρ where $X(-\rho)$ is a subjective gain, $X(\rho)$ is a larger gain and $v'(X(-\rho)) > v'(X(\rho)) > 0$ because v is strictly increasing and concave over gains. For those values of ρ where $X(-\rho)$ is a subjective loss, $-X(-\rho)$ is a gain and $v'(X(-\rho)) > v'(-X(-\rho)) > v'(X(\rho)) > 0$. The first inequality is true because v is strongly loss averse. The second and third inequalities are true because $X(\rho) > -X(-\rho)$ if the portfolio's mean return is a subjective gain and v' is positive and decreasing over gains.

In particular, probability weighting will tend to increase the market price of risk as it emphasizes the extreme outcomes, making investors more reluctant to take on the risk of the market. Loss aversion will also do this, though the convexity of utility for losses will tend to offset a higher price of risk. Furthermore, the makeup of the market portfolio itself will typically change, as a different market price of risk will alter the point of tangency of the borrowing-lending line.

Although this proposition is labeled as a CPT CAPM, it is applicable in a much wider context. Utility can be globally risk averse, S-shaped, a combination of the two, as in Bowman *et al.* (1999), or have other shapes. It need only be strictly increasing and sufficiently well behaved that expected utility is defined. The probability weighting function need not emphasize rare events or have any particular shape provided $\Omega' > 0$ so that it always assigns positive decision weights. The assumptions of strong loss aversion and a CSQW, which are used in Proposition 10 to prove that variance is disliked, are not required for the CAPM. Proposition 11 proves that if any equilibrium exists, then it must be the objective CAPM simply because portfolios with higher means are always preferred.

However, the proposition does not give explicit conditions for the existence of an equilibrium. This requires that each investor's demand be finite, which necessitates that the indifference curves become steeper than the slope of the borrowing-lending line for sufficiently high σ so that they have a finite tangency.[46] In the standard, risk-averse model, indifference curves are increasing and convex, so buying and borrowing pressure in an exchange economy will increase stock prices or the interest rate thereby reducing the slope of the borrowing-lending line until markets clear. But as shown in Proposition 10, a dislike of variance only guarantees that indifference curves are increasing not that they are convex. Other conditions are needed to ensure that the demand for leverage is finite.

One simple way to guarantee finite leverage is an explicit portfolio constraint like a maximum tolerable loss. For the realization, ρ, the

[46]There must also be nonzero demand for the tangency portfolio. This is also true in the standard model, and the sufficient condition here is the same. If the tangency portfolio has an expected rate of return μ_t and standard deviation σ_t, a levered position has a return of $r_f + b(\mu_t - r_f) + b\sigma_t\tilde{\rho}$. The change in expected utility due to increasing leverage starting from an initial position of no leverage is $(\partial \mathbb{E}_\Omega[v]/\partial b)|_{b=0} = \mathbb{E}_\Omega[v'(r_f - \hat{x})(\mu_t - r_f + \sigma_t\tilde{\rho})] = v'(r_f - \hat{x})(\mu_t - r_f)$ so every investor's demand will be positive if the tangency portfolio's expected rate of return exceeds the risk-free rate, assuming v is differentiable.

subjective rate of return on the tangency portfolio levered by b is $r_f + b(\mu_t - r_f + \sigma_t \rho) - \hat{x}$. Therefore, to ensure a minimum return of x_0 for bounded elliptic variables with $|\rho| \le a$, leverage must therefore be limited to[47]

$$b \le \frac{x_0 - r_f + \hat{x}}{\mu_t - r_f - \sigma_t a} \quad \text{if} \quad \frac{\mu_t - r_f}{\sigma_t} < a. \tag{33}$$

If the Sharpe ratio is higher than a, there is no restriction on leverage as the worst return improves with b. However, if this is true and some investor does desire an infinite position, his demand will increase stock prices or the interest rate, lowering the Sharpe ratio until an equilibrium is achieved, just as in the standard model.[48]

If the elliptic variable has an unbounded domain, then a finite worst return is not possible on any portfolio other than the risk-free asset alone so any constraint must be probabilistic; e.g., the probability that the return falls short of x_0 must be smaller than p_0. The probability of a return smaller than $r_f + b(\mu_t - r_f + \sigma_t \rho) - \hat{x}$on the levered tangency portfolio is $F(\rho)$. So, assuming $p_0 \le 1/2$,[49] leverage must be limited by

$$b \le \frac{x_0 - r_f + \hat{x}}{\mu - r_f + \sigma F^{-1}(p_0)} \quad \text{if} \quad \frac{\mu - r_f}{\sigma} < -F^{-1}(p_0). \tag{34}$$

Here $-F^{-1}(p_0)$ replaces a in (33), but otherwise the reasoning remains the same.

In the absence of some explicit portfolio constraint, there must be restrictions on the utility and weighting functions to ensure finite demand.

[47]If no risk-free asset exists, then this and later results can be handled using the zero-beta version of the CAPM. By the usual mean-variance mathematics, there is a minimum-variance zero-beta portfolio corresponding to any efficient portfolio on the upper limb of the minimum variance hyperbola. A portfolio combination of these two with fraction b invested in the efficient portfolio has a return of $x = \mu_z + b(\mu_t - \mu_z) + [b^2\sigma_t^2 + (1-b)^2\sigma_z^2]^{1/2}\rho - \hat{x}$ where μ_z and σ_z are the mean and standard deviation of the zero-beta portfolio and ρ is a standard elliptical variable. Because the square root factor is larger than $b\sigma_t$, the portfolio's return is less than $\mu_z + b(\mu - \mu_z + \sigma_t \rho) - \hat{x}$ when $\rho < 0$ and the constraint would be violated. Therefore, the limitations on leverage as given in (34) and (35) with r_f replaced by μ_z apply. In fact, this constraint on leverage is too loose. Weaker restrictions are sufficient to show that unbounded leverage is undesirable.

[48]This does not mean the Sharpe ratio cannot exceed a in the resulting equilibrium; a is simply the largest value ensuring an equilibrium regardless of preferences.

[49]Deriving the inequality requires that the denominator in (35) is negative; therefore, $F^{-1}(p_0)$ must be as well. If the restriction is instead that the subjective probability weight cannot exceed P_0, then $F^{-1}(\Omega^{-1}(p_0))$ replaces $F^{-1}(p_0)$ in Equation (35).

From Proposition 2, investors with extreme-risk avoidance who use the true probabilities do not take unbounded positions.[50] This result extends to investors who use CSQW decision weights, as they increase the importance of the lower tail and therefore reduce expected utility. But these assumptions are stronger than needed.

Barberis and Huang (2008) considered a related case with a multivariate normal distribution and investors having identical TK utility (with $\alpha = \beta$), identical TK weighting functions, and a zero-utility reference return equal to the risk-free rate. They show that an equilibrium exists. On the other hand, De Giorgi *et al.* (2004) show that an equilibrium does not exist if the model is altered to have investors with heterogeneous TK utility functions each with $\alpha_i = \beta_i$.[51] There is no equilibrium precisely because such investors lack extreme-risk avoidance and desire unbounded positions in the tangency portfolio. To ensure an equilibrium, the latter authors propose a piecewise exponential utility function that is bounded below and above and also ignore probability weighting.[52] Together, these assumptions are operationally equivalent to extreme-risk avoidance and ensure that high leverage is not desirable.

This result is easily extended to cover all cases with bounded, weakly loss-averse utility functions with $\hat{x} \leq r_f$ and probability weighting functions characterized by $\Omega(1/2) \geq 1/2$. Like the concavity of a CSQW, the latter assumption is a form of pessimism generating a subjective median below the objective median. Suppose the limiting values of the utility function are $v(-\infty) = \underline{v}$ and $v(\infty) = \bar{v}$, with $-\underline{v} > \bar{v}$ because utility is weakly loss averse.[53] The zero-utility point for a levered position in the tangency portfolio occurs at $\rho = (\hat{x} - r_f)/b - S$ where S is the tangency portfolio's Sharpe ratio. In the limit as $b \to \infty$, subjective losses

[50] Proposition 2 was proved for a finite state space, but it can be extended to a continuous state space with elliptical assets by a limiting argument.

[51] Some heterogeneity is typically required for an equilibrium to fail to exist. If all investors are identical, then there usually is a trivial equilibrium with each investor holding exactly the market portfolio since that is the only symmetric feasible strategy. This simplification is, of course, the basis for representative agent models.

[52] Even bounded utility is insufficient to guarantee the existence of an equilibrium with no restrictions on the probability weighting function.

[53] Since utility is strictly increasing and bounded, the greatest lower bound and the least upper bound of the utility function are the limiting values. This proof remains valid if there is no lower bound on utility, but if there is only an upper bound, then utility has extreme-risk avoidance and Proposition 2 applies.

(gains) occur for all realizations of ρ less (greater) than $-S$. In addition, all subjective losses will have a realized utility of $\underline{v}$, and all subjective gains will have a realized utility of $\bar{v}$ except at the singular point of zero utility, which occurs with probability zero. In the limit, expected utility is therefore

$$\mathbb{E}_\Omega[v(r_f+b(\mu-r_f)+b\sigma\tilde{\rho}-\hat{x})] \xrightarrow[b\to\infty]{} \Omega(F(-S))\underline{v}+[1-\Omega(F(-S))]\bar{v}. \quad (35)$$

If S is zero, then expected utility is negative because $\Omega(F(0)) \geq 1/2$ by the assumed pessimism. By continuity, the expected utility remains negative for Sharpe ratios near zero. So, as before, stock prices are always bid up to keep demand finite at some positive Sharpe ratio and an equilibrium will obtain. If $\Omega(1/2) < 1/2$, there may be no equilibrium as demand for the tangency portfolio might be infinite even with a Sharpe ratio of zero, and, of course, with a negative Sharpe ratio, all investors will wish to short it.[54]

Unfortunately, while elliptical returns might be a reasonable description of the returns on the primary assets in an economy, it certainly does not describe the returns on the myriad derivative contracts that could be introduced on those primary assets. In fact, if a set of assets has an elliptical return, then a vanilla call or put option on any one of them *cannot* have a return that falls into the same elliptical class because one tail of the distribution is eliminated. Fortunately, the CAPM equilibrium can still be partially valid even if two-fund separation does not hold.

Dybvig and Ingersoll (1982) have shown that the linear relation between risk premia and beta holds for the set of primary assets described by an elliptical distribution even in the presence of non-elliptical derivative contracts, provided the market is complete or effectively so. The same analysis applies here to the objective moments so only a summary of the reasoning is given. The elliptical distributions fall within the class of Ross' separating distributions so amongst just the primary assets any portfolio that is not mean-variance efficient is second-order stochastically dominated. From Proposition 6, the market portfolio is objectively risk-averse efficient within a broader class so it cannot be stochastically dominated, and it must, therefore, be mean-variance efficient within just the primary assets.

[54]This equilibrium-existence result does not require any special properties for investors' probability weighting functions other than the "pessimism" assumption that $\Omega(1/2) \geq 1/2$. Consequently, it remains valid if separate gain and loss weighting functions are employed provided only that $\Omega^-(1/2) > \Omega^+(1/2)$.

This reasoning does not mean that the CAPM can be used to price derivatives. Only the primary (elliptically distributed) assets need have a linear relation between their risk premiums and market betas.

8 Conclusion

This paper has analyzed the optimal portfolios of investors who have CPT's S-shaped utility or use probability weighting to assess outcomes. There is a primary difficulty inherent in each of these two assumptions. S-shaped utility induces a partial preference for risk and can lead to unbounded demand resulting in a failure of equilibrium. Cumulative probability weighting produces decision weights that depend on the outcome's order, which, in any portfolio problem, are endogenously determined. This increases the complexity of the problem, but, more importantly, it can destroy results like the efficiency of the market portfolio in a complete market, which depend on identical orderings.

The notion of extreme-risk aversion, defined here, or a maximum tolerable loss eliminate portfolios with unbounded positions and restore most of the intuition about optimal portfolios that is present in models with risk aversion. In addition, models with equally probable states (including models with a continuous state space) reestablish the comonotonicity property of optimal portfolios and restore the existence of a representative (average) investor.

Under CPT preferences, Ross' two-fund separation result holds only for a subset of symmetric probability distributions and requires, in addition, a limited set of concavified symmetric Quiggin weighting functions. However, the CAPM pricing result of expected returns linear in beta, derived under the assumption of elliptical distributions, holds whenever an equilibrium exists.

The standard risk-averse representative-agent pricing results, including the special case of the CAPM, continue to be valid in CPT. Therefore, the distinct predictions of that theory are likely to be ones of degree and not substance. The numbers or calibration will change, but not the basics. Of course in some other contexts even those may change.

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